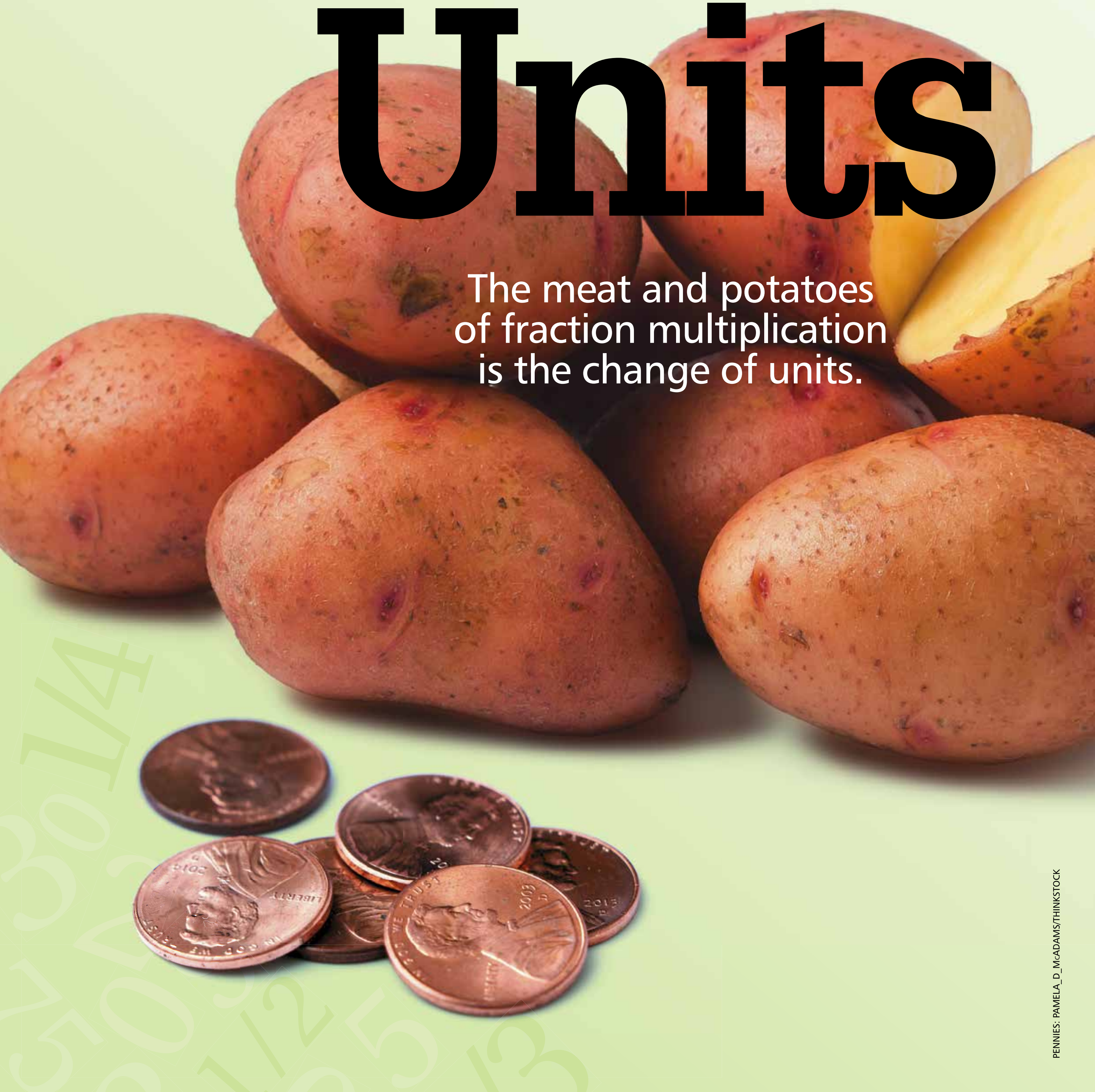
Units

The meat and potatoes of fraction multiplication is the change of units.



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Matter

POTATOES: ANDREY EREMIN/THINKSTOCK

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and Zaur Berkaliev

Imagine a fifth-grade classroom where the teacher is trying to link multiplication of fractions to multiplication skills that students have previously learned.

Teacher: What is four times three?

Student: Three plus three plus three plus three is twelve.

Teacher: Can you do two times one-third?

Student: Yes. One-third plus one-third is two-thirds.

Teacher: Let's do one more. What is two-thirds times five-sevenths?

Student: Uh . . . .

Students are confused when the teacher moves to multiplying a fraction times a fraction because they realize that their solution strategy, which was successful for the other two questions, does not work for the last one. What is wrong? All three questions asked students to multiply, and students used repeated addition as a way to find the product. But in the case of a fraction times a fraction, the repeated addition strategy fails. No clear cognitive bridge emerges between the repeated addition strategy and the problem of multiplying $2/3 \times 5/7$. When the teacher recognizes that repeated addition is not an intuitive approach to this problem, she typically introduces a new procedure to students: Multiply the two denominators to get the new denominator, and multiply the two numerators to get the new numerator. However, this does not accomplish the teacher's initial intention of connecting to students' prior knowledge.

Is there any way to describe multiplication for all real numbers so that students are able to use their prior knowledge about whole-number multiplication for learning a new case of multiplication with fractions? How can we initially teach the general concept of multiplication so to avoid confusing our students when we introduce other number types?

Whole-number multiplication models in textbooks and standards

To determine the progressions of teaching multiplication, we examined two basal textbooks that are widely used in the United States: Scott Foresman-Addison Wesley (Charles et al. 2005) and Investigations (Pearson Scott Foresman TERC 2008) from grades 3 to grade 5. In both textbooks, multiplication officially begins in grade 3 with such problems as these:

> Three brushes are in each glass jar. How many brushes are there if you have 5 glass jars?" (Charles et al. 2005)

> Here are 4 stars. Each star has 5 points. How many total points are in 4 stars? (Pearson Scott Foresman TERC 2008)

The problems above imply the concept of equal-size groups, a fundamental condition in defining multiplication. Overall, we could see three models of whole-number multiplication in these textbooks:

1. Equal-size groups
2. Arrays
3. Repeated addition

However, the distinction among these models is unclear because each row or column of the array model can be seen as an equal-size group, and repeated addition can describe the other two models. All models used in both textbooks include only discrete objects, such as flowers, apples, or shapes. We did not find any use of continuous quantities, such as length of a string, area of a floor, or volume of water. Discrete models work fine when dealing with whole numbers, but using only this type of model might cause difficulty when students learn multiplication of rational numbers or real numbers—as we saw in the opening vignette.

The Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010) suggests three types of multiplication and division situations:

1. Equal group
2. Array
3. Compare

Each can be modeled with discrete objects as well as with such continuous quantities as measurement or area. Although both contexts are important, the following paragraph shows the significance of using continuous models.

> Counting discrete items often convinces students that the size of things counted does not matter (there could be exactly 10 toys, even if they are different sizes). In contrast, for measurement, unit size is critical, so teachers are advised to plan experiences and reflections on the use of other units and length-units in various discrete counting and measurement contexts. Given that counting discrete items often correctly teaches students that the length-unit size does not matter, so teachers are advised to plan experiences and reflections on the use of units in various discrete counting and measurement contexts. (The Common Core State Standards Writing Team 2012, p. 13)

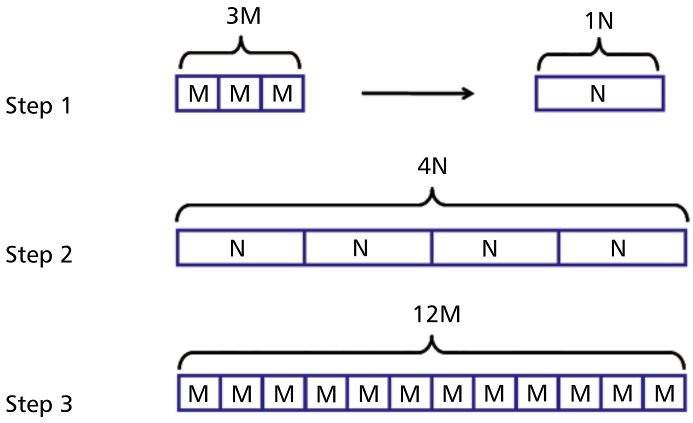
A unit cannot be separated from a number, and a number is dependent on its unit.

The concept of multiplication

Because the focus is often on performing a multiplication algorithm or recalling a basic multiplication fact, the concept of multiplication may sound strange. When students can compute multiplication problems accurately, we often assume that they "understand" multiplication. However, if students do not have a solid concept of multiplication, they may be unable to make connections to advanced

FIGURE 1

A unit cannot be separated from a number, and a number is dependent on its unit. Below is an illustration of the relationship between a unit and a quantity in multiplication.



mathematical concepts, such as ratio, slope, or rate of change (NCTM 2011). Furthermore, CCSSM supports the distinction between multiplicative reasoning and additive reasoning (4.OA.A2, p. 29). The distinguishable characteristic between additive and multiplicative reasoning is *the change of unit of measure* (Berkaliev 2008). Similarly, the Common Core State Standards Progression draft, K–grade 5 Geometric Measurement (The Common Core Standards Writing Team 2012) uses "units of units" to describe the equal-size group model of multiplication. We believe this phrase, *units of units*, clearly illustrates the essential nature of multiplication, which is discussed in the following section.

The meaning of unit

We use units in everyday life. In a grocery store, we purchase 500-milliliter water bottles and 2 pounds of potatoes, and we pay 12 dollars 28 cents. Milliliter, pound, dollar, and cent are all units. If you change one unit to another—for example, milliliter to liter—the number that represents the amount of water, 500, will be changed to 0.5, or one-half of a liter. A unit cannot be separated from a number, and a number is dependent on its unit. This explicit use of unit is usually observed in scientific measurement activities, but units are also used implicitly in mathematics.

In the measurement perspective, all numbers involve the concept of unit. Berkaliev (2008) indicated that "a number itself is not just an absolute and final entity, but is only an expression of a relationship between two different quantities" (p. 9). When multiplying two numbers, one number is the unit of measure and the other is a quantity to be measured (see **fig. 1**). For example, if a quantity *M* is iterated three times to create intermediate unit *N*, and unit *N* is iterated four times to create a new quantity, then we can say that 3 × 4 = 12, using unit *M* as the initial unit (see **fig. 1**) (Dougherty and Venenciano 2007). This illustrates the relationship between the initial unit (*M*) and the product represented by quantity (4*N*) in whole-number multiplication. It also exemplifies the difference between addition and multiplication: Addends in addition simply use one common unit, and multiplication involves more than one unit. For example, 4 × 3 means four of three units or four groups of three units. As the first step of this multiplication, we must consider the iteration of unit *M* as a newly created unit. The second step is to find four of the new unit; in that case, we can find it by iterating the new unit four times because the multiplier is a whole number. Finally, we go back to the original unit to measure the final quantity, and then we will conclude that the product is to be twelve units. The new unit is called an *intermediate unit* because it bridges two different quantities (Dougherty and Venenciano 2007).

Third graders, as part of the Measure Up project (Dougherty 2008), created an intermediate unit when given the task of re-creating a volume quantity. The teacher, Ms. Z, showed them a large container of water that could not be moved. They were asked to re-create the same volume of water using a small container (volume unit). After measuring the original quantity with the small unit, Reed suggested, "We should make another unit that would be bigger so it won't take so long."

Ms. Z asked, "How could we make another unit?"

Richard responded, "Well, we could use, like, eight of the small ones to make a bigger one. Then we could use that."

"But how would we know how many of the original units it took?" Ms. Z questioned.

Macy explained, "Every time you use the new unit, you are really using eight of the small ones."

In this short interchange, with no introductory instruction, the use of an intermediate unit was a natural way to approach the task. And, by using volume as the context for the

FIGURE 2

Seeing (a) one row or (b) one column as an intermediate unit in an array model is natural; but (c) various other shapes are possible, too.

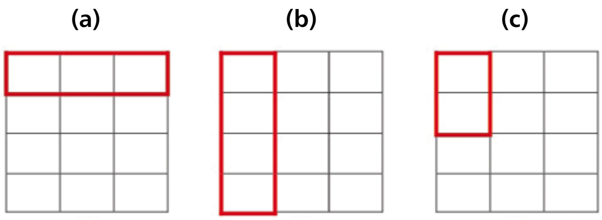


FIGURE 3

The change of unit in multiplying 3/4 × 3 aligns closely with the whole-number model in **figure 1**.

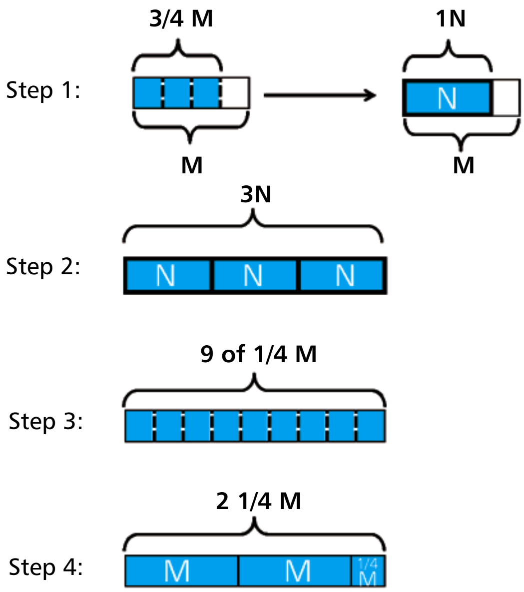
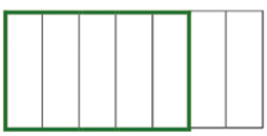
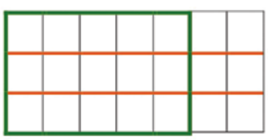


FIGURE 4

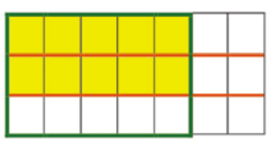
As with whole-number multiplication, the initial unit changes in an array model of multiplying fractions. Textbooks often use this model—*without* the link to an intermediate unit.



(a) Find 5/7 of the whole unit (the largest rectangle). 5/7 acts as an intermediate unit.



(b) Divide the intermediate unit into three equal parts. A row divided by orange-colored lines represents 1/3 of the intermediate unit.



(c) Take 2 rows for 2/3 of the intermediate unit, which is 2/3 of 5/7 of the whole unit. The smallest rectangle denotes 1/21 of the whole unit, so the shaded quantity is 10/21.

problem, students saw one (intermediate) unit but understood that it represented eight of the original units.

We can perceive the intermediate unit in the array model as well. Similar to the equal-size group model, the array model clearly implies the idea of intermediate unit. It is natural and convenient to see one row or one column as an intermediate unit, although designating other shapes as an intermediate unit is possible (see **fig. 2**).

Although we use arrays of the same size in **figure 2**, each intermediate unit represents a different multiplication situation. When we take a row of three units as an intermediate unit (see **fig. 2a**), the array model represents 4 × 3. When we choose a column of four units (see **fig. 2b**) as the intermediate unit, the model corresponds to 3 × 4. If we assume that the two-unit shape in **figure 2c** is an intermediate unit, the multiplication represented here is 6 × 2 because the intermediate unit can be iterated six times to measure the array. Other representations are possible by using a one-unit, twelve-unit, or six-unit intermediate unit. Both the equal-size group model and the array model of multiplication imply the idea of intermediate unit, but it is important for teachers to emphasize how an intermediate unit exists and works in multiplication and how it differs significantly from addition.

Applying intermediate units for nonwhole numbers

The concept of intermediate unit fits well with multiplication of rational numbers. If the multiplication involves a fraction times a whole number, such as 3/4 × 3 (see **fig. 3**), the intermediate unit is partitioned to show 3/4 (see step 1). Then the intermediate unit is iterated three times (see step 2). Thus the final product is 9/4 (see step 3), or 2 1/4 (see step 4). Using an intermediate unit with this multiplication situation closely aligns with the whole-number model (see **fig. 1**).

What if the multiplication involves two fractions, such as 2/3 × 5/7? The intermediate unit of this multiplication is 5/7. Thus, we need to find 2/3 of the intermediate unit. An array model (see **fig. 4**) explains this multiplicative situation more effectively than the equal-size group model because partitioning a quantity multiple times is convenient in this model. As with whole-number multiplication, the initial unit is changed (see **figs. 4a** and **4c**). The first step (**4a**) is to determine the intermediate unit. Next, we consider the quantity as a whole unit to find 2/3 of it (**4b** and **c**). Finally, we go back to the original whole unit to represent the final quantity (**4c**). We can see that each row represents 1/3 of the whole unit and that each column represents 1/7 of the whole unit. The size of the smallest rectangle is read as 1/7 of a row, 1/3 of a column, or 1/21 of the whole unit. Hence, we have ten 1/21's of the whole unit at the end, which makes our final answer 10/21. This model is often used in textbooks without the link to an intermediate unit.

A similar process can be applied with other models of multiplication. But if the model contains only discrete items, representing rational numbers that include partial units is impossible. Moreover, explaining multiplication of fractions will be quite difficult, as will representing fractions in the array model if it is used in a discrete context.

Focus on fundamental conceptual understanding

Learning multiplication is a greater challenge for students in early grades than learning addition or subtraction. Studies (Sieman 2004; Sudarshan and Aye 2008) have revealed that teachers tend to focus on procedural knowledge rather than conceptual understanding when they teach multiplication to students in the early grades. Students must learn more from multiplication than memorizing facts or mere calculation. They need to develop multiplicative reasoning, which is fundamental to accessing mathematics that is more sophisticated and complex. When multiplying two bare numbers or memorizing facts without developing an understanding of multiplicative

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situations, students are likely to narrow their thinking and be captured by misconceptions on multiplication (Smith and Smith 2006).

Young children spend a much greater amount of time on practicing multiplication facts as compared with understanding the concept of multiplication. When students have long-term, foundational concepts rather than a series of fragmented algorithms or facts, they are more likely to understand and generalize the mathematics. Using generalized models that represent both concepts and procedures is an important part of students' development of mathematical understanding. We claim that the concept of unit and intermediate unit has the potential to establish a general model of multiplication and that the heart of multiplication is the change of units. When students have sufficient knowledge of the general concept of multiplication, we believe they can have a better sense of multiplication of all real numbers.

Common Core Connections
5.NF.4

REFERENCES

Berkaliev, Zaur. 2008. "Generalized Properties of Real Numbers in the Elkonin-Davydov Curriculum as Addressed through Vygotsky's Concept of Ascending to the Concrete." Presentation at the National Council of Teachers of Mathematics Research Conference in Salt Lake City, UT.

Charles, Randall I. 2005. *Mathematics Grade 3*. Glenview, IL: Scott Foresman-Addison Wesley.

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics (CCSSM). Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

Common Core State Standards Writing Team. 2012. "Progressions for the Common Core State Standards in Mathematics: 6–7, Ratios and Proportional Relationships." http://ime.math.arizona.edu/progressions/

Dougherty, Barbara J. 2008. "Measure Up: A Quantitative View of Early Algebra." In *Algebra in the Early Grades*, edited by James Kaput, David W. Carraher, and Maria L. Blanton, pp. 389–412. Mahweh, NJ: Lawrence Erlbaum Associates.

Dougherty, Barbara J., and Linda Venenciano. 2007. "Measure Up for Understanding: Reflect and Discuss." *Teaching Children Mathematics* 13 (May): 452–56.

Otto, Albert Dean, Janet H. Caldwell, Cheryl Ann Lubinski, and Sarah Wallus Hancock. 2011. *Developing Essential Understanding of Multiplication and Division for Teaching Mathematics in Grades 3–5*. Essential Understanding Series. Reston, VA: National Council of Teachers of Mathematics.

Pearson Scott Foresman TERC. 2008. *Investigations in Number, Data, and Space. Grade 3*. Glenview, IL: Pearson Scott Foresman TERC.

Sieman, Dianne. 2004. "Partitioning—the Missing Link in Building Fraction Knowledge and Confidence." Adelaide, Australia. http://www.aamt.edu.au/Library/Other/Fractions/Partitioning-the-missing-link/(language)/eng-AU

Smith, Stephanie Z., and Marvin E. Smith. 2006. "Assessing Elementary Understanding of Multiplication Concepts." *School Science and Mathematics* 106 (March): 140–49.

Sudarshan, Akhila, and Khin Maung Aye. 2008. "Teacher Practices in Mathematics Classrooms with At-Risk Students." Paper presented at the Australian Association for Research in Education Annual Conference, Brisbane, Australia.

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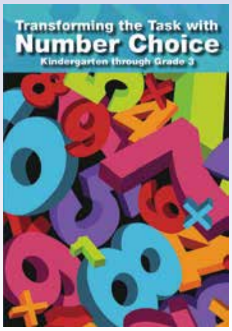

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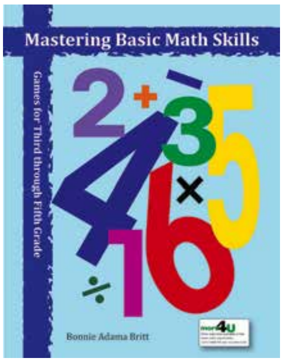

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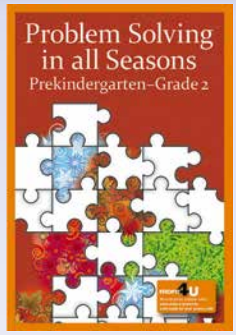

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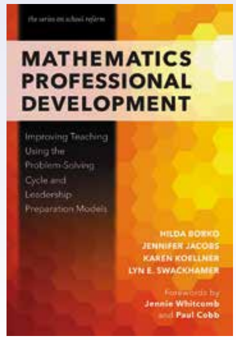

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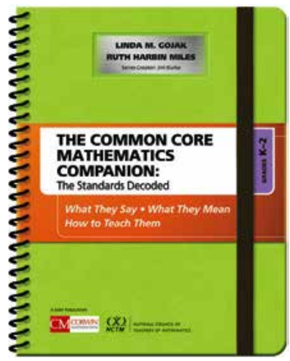

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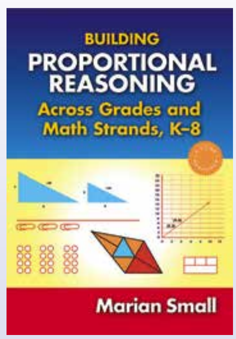

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