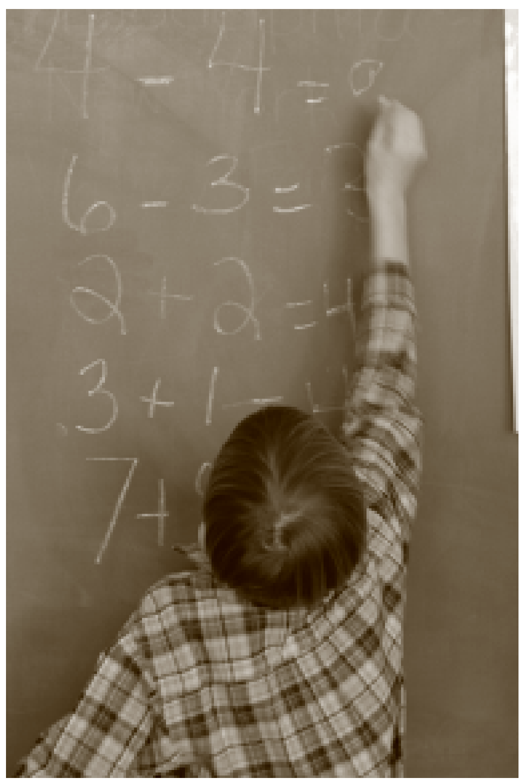
Using Research to Shift From the "Yesterday" Mind to the "Tomorrow" Mind

Teaching and Learning Mathematics



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March 2000

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About This Document

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Abbreviations

EALRs	Essential Academic Learning Requirements
WASL	Washington Assessment of Student Learning

Chapter 1

INTRODUCTION

Welcome! In as friendly and useful manner as possible, our goal is to provide a research-based overview of the potential and challenges of teaching quality mathematics (K–12). Though the primary contexts are the Washington State essential academic learning requirements (EALRs) in mathematics and the correlated Washington Assessment of Student Learning (WASL), each reader must interpret and reflect on the content within his/her own district or classroom situation. Without this important step toward interpretation and reflection by each reader, this publication becomes yet one more resource to be piled on a shelf for reading on that rainy day that never seems to come in Washington.

We are fully aware of the ominous nature of the word "research" and its associated baggage. The mere inclusion of the word in the title of articles or workshop offerings often causes teachers and administrators to look for an escape route, whether it is physical or mental. Yet, our intent is to counter this attitude by constructing a research-based perspective that helps both teachers and administrators further the mathematics education reform efforts in Washington at all grade levels. As Charles Kettering, an American engineer and inventor (1876–1958), once said:

> Research is a high-hat word that scares a lot of people. It needn't. It is rather simple. Essentially, research is nothing but a state of mind–a friendly, welcoming attitude toward change … going out to look for change instead of waiting for it to come. Research … is an effort to do things better and not to be caught asleep at the switch…. It is the problem solving mind as contrasted with the let-well-enough-alone mind…. It is the "tomorrow" mind instead of the "yesterday" mind.

From Kettering's words, we pull the guiding theme for this book: to use research-based information to support the necessary shift from a "yesterday" mind to a "tomorrow" mind in the making of the many decisions as to how mathematics is taught or learned in Washington.

An introductory road map through this text can be useful, especially for those readersreluctant to make the trip.

- First, we discuss what research in mathematics education can and cannot do. This section is important because it helps orient the "tomorrow" mind in a positive direction while also ensuring that teachers and administrators are aware of potential misuses of research.

- Second, we overview some of the research results related to each of the essential learning academic requirements in mathematics. The key word here is "some," as the volume of research available in mathematics education is quite large and varied (in both quality and applicability). Though an attempt was made to sort through and select research results in a fair manner, the goal of supporting the "tomorrow" mind was always in full view. If we omitted mention of research results that you have found useful, we apologize for their omission and suggest that you share them with your colleagues. Also, some of the research results mentioned may seem dated but was included because it contributes in some fashion to our current situation and concerns.

- Third, we address specific questions in mathematics education as raised by teachers, administrators, or parents. These questions range from the classroom use of calculators or manipulatives, to the role of drill and algorithmic practice, to the best models for the professional development of teachers. In most instances, the research evidence is not sufficient to answer the questions raised in a definitive manner. We suggest that even small insights or understandings are better than teaching in the dark.

- And fourth, we outline a plan that a teacher, district, or state can follow to maintain relevancy relative to this document and the issues it addresses. That is, the text should be viewed as another small step forward for Washington State teachers and administrators. Combined with the other efforts of the Office of Superintendent of Public Instruction (OSPI), districts, administrators, teachers, and professional groups, these steps forward help us both gain and maintain momentum in adopting the "tomorrow" mind in mathematics education.

Given that road map, we ask you to now join us on this trip through the field of mathematics education research and hope that you find the journey useful. As our ultimate goal is to support teachers and administrators in their efforts to improve student learning in mathematics, we know that an increased awareness of research results is an important form of support. Our apologies are offered if we have either misrepresented or misinterpreted the research results as reported. Also, we apologize in advance for any misleading interpretations or summaries of the research conclusions of others; these lapses were not intentional.

RESEARCH IN MATHEMATICS EDUCATION: WHAT IT CAN AND CANNOT DO

Think of the many things that can be investigated in mathematics education; it is easy to be overwhelmed. Four key ingredients can be identified:

- The students trying to learn mathematics—their maturity, their intellectual ability, their past experiences and performances in mathematics, their preferred learning styles, their attitude toward mathematics, and their social adjustment.

- The teachers trying to teach mathematics—their own understanding of mathematics, their beliefs relative to both mathematics itself and how it is learned, their preferred styles of instruction and interaction with students, their views on the role of assessment, their professionalism, and their effectiveness as a teacher of mathematics

- The content of mathematics and its organization into a curriculum—its difficulty level, its scope and position in possible sequences, its required prerequisite knowledge, and its separation into skills, concepts, and contextual applications.

- The pedagogical models for presenting and experiencing this mathematical content—the use of optimal instructional techniques, the design of instructional materials, the use of multimedia and computing technologies, the use of manipulatives, the use of classroom grouping schemes, the influences of learning psychology, teacher requirements, the role of parents and significant others, and the integration of alternative assessment techniques.

All of these ingredients, and their interactions, need to be investigated by careful research. Again, it is easy to be overwhelmed (Begle and Gibb, 1980).

Our position is that educational research cannot take into account all of these variables. The result we must live with is acceptance that educational research cannot answer definitively all of the questions we might ask about mathematics education. At best, we can expect research in mathematics education to be helpful in these ways:

- It can inform us (e.g., about new pedagogical or assessment techniques).
- It can educate us (e.g., about the pros/cons of using different grouping models).
- It can answer questions (e.g., about the potential impact of professional development models for teachers).
- It can prompt new questions (e.g., about the impact of using the Internet to make real-world connections).
- It can create reflection and discussion (e.g., about the beliefs that students and teachers hold toward mathematics).
- It can challenge what we currently do as educators (e.g., about our programs for accommodating students with differing ability levels or learning styles).
- It can clarify educational situations (e.g., about how assessment can inform instruction).
- It can help make educational decisions and educational policy (e.g., about student access to calculators or performance benchmarks).

Yet, research in mathematics education can also be counterproductive or fall short of what we would expect in these ways:

- It can confuse situations (e.g., about which math curriculum is the best).
- It can focus on everything but your situation (e.g., about your classroom, your specific students, and their learning of mathematics).
- It can be hidden by its own publication style (e.g., its scholarly vocabulary and overwhelming statistics).
- It can be flawed (e.g., about the interpretation of the research data).
- It can be boring and obtuse (e.g., its technical jargon, its overuse of statistics and graphs, and its pompous style).

Above all, despite the wishes of many teachers and administrators, educational research cannot PROVE anything! At best, educational research provides information that the community of educators can use, misuse, or refuse.

It is a well-established notion that research results tend not to be used by educators and at times are purposely ignored. For example, Reys and Yeager (1974) determined that while 97.5 percent of the elementary teachers frequently read general education journals, 87.5 percent of these same teachers seldom or never read the research-flavored articles. When asked why, 80 percent of the teachers replied with "lack of time" or "lack of direct classroom implications." In contrast and on a more positive side, Short and Szabo (1974) found that mathematics teachers at the secondary level were much more knowledgeable about and favorable toward educational research than their colleagues in English and social science.

The situation needs to change, as research results must be reflected on and integrated as an important part of the mathematics education plan and process in Washington. The entire education community—mathematics teachers, administrators, legislators, parents, and college mathematics educators—must take and share in the responsibility for this reflection process and integration of research results, whether it occurs at the individual learner level, the classroom level, the district level, the university level, or the state level. This resource text is designed to serve as a catalyst for promoting reflection, discussion, and problem solving within this education community, helping this same community continue to shift from the "yesterday" mind to the "tomorrow" mind in its approach to mathematics education.

Chapter 2

OVERVIEW OF THE RESEARCH: A WASHINGTON STATE PERSPECTIVE

Each of the EALRs will be considered within the context of some known research results. The search for relevant research results was broad but not exhaustive. The majority of the research results have been omitted (fortunately or unfortunately for the reader), with the few results selected being those that can whet your appetite and illustrate best how research can inform and educate the education community. When it was appropriate, research results that are conflicting or complementary have been juxtaposed to prompt further reflection and discussion.

The text's format will vary from the reporting of interesting research results to the suggesting of research implications that can be adapted for use within a classroom. Without being obtrusive, references are included for those readers who would like to pursue the ideas in more detail.

Our primary constraint was to provide summaries of research results in a very concise format. In most instances, this constraint precluded any attempts to describe the actual research that was done. Thus, we often had to omit important factors such as the subjects' ages, the subjects' grade levels, the population size, the experimental design, the null hypotheses, the experimental instruments, the data analysis, the levels of statistical significance, or the researchers' interpretations. These omissions can be dangerous, as it may be misleading to state conclusions based on research involving a few subjects and without replication. Furthermore, no formal effort was made to evaluate the quality of the research efforts as criteria for inclusion in this text. As such a broad review by itself is enormous, we now leave it to you the reader to investigate further each result and evaluate its reasonableness.

RESEARCH ON NUMBER SENSE

Number and Numeration

- The research is inconclusive as to a prerequisite relationship between **number conservation** and a child's ability to learn or do mathematics (Hiebert, 1981).

- A child's acquisition of and fluency with the number-word sequence (e.g., one, two, three …) is a **primary prerequisite for the ability to count.** A worthy goal is for the student's fluency to be bidirectional, where the number-words can be produced in sequence in either direction easily (Bergeron and Herscovics, 1990).

- Colored chips and money often are used as manipulatives to represent **place value concepts and operations,** but they prompt increased cognitive complexity. The reason is that the place value notions are not explicitly represented in the color of the chips or the physical sizes of the money (English and Halford, 1995).

- **Place value** is extremely significant in mathematical learning, yet students tend to neither acquire an adequate understanding of place value nor apply their understanding of place value when working with computational algorithms (Fuson, 1990; Jones and Thornton, 1989).

- A major reason for **place value lapses** is the linguistic complexity of our place-value system in English. For example, we do not name "tens" as done in some languages (e.g., "sixty" vs. "six-tens"), arbitrarily reverse the number names between 10 and 20 (e.g., eleven and thirteen), and accept irregularities in our decade names (e.g., "twenty" vs. "sixty") (Fuson, 1990; English and Halford, 1995).

- Students confronted with a new written symbol system such as **decimals** need to engage in activities (e.g., using base-ten blocks) that help construct meaningful relationships. The key is to build bridges between the new decimal symbols and other representational systems (e.g., whole number place values and fractions) before "searching for patterns within the new symbol system or practicing procedures" such as computations with decimals (Hiebert, 1988; Mason, 1987).

- In their extensive study of student **understanding of place value,** Bednarz and Janvier (1982) concluded that:

    1. Students associate the place-value meanings of "hundreds, tens, ones" more in terms of order in placement than in base-ten groupings.
    2. Students interpret the meaning of borrowing as "crossing out a digit, taking one away, and adjoining one to the next digit," not as a means of regrouping.

- Students often fail to make the correct interpretation when **using base-ten blocks to model place-value or an addition computation.** They might not arrange the blocks in accordance with our base-ten positional notation (decreasing value left-to-right) or they might manipulate the blocks in any order (trading whenever necessary or adding left-to-right in place values). Teachers need to be aware that both of these possibilities occur as natural events when students use base-ten blocks (Hiebert, 1992).

- **Rational number sense** differs from whole number sense. The primary difference seems to be that rational number sense is directly connected to students' understanding of decimal and fraction notations, while whole number sense does not have to be directly connected to the written symbols (Sowder and Schappelle, 1989; Carraher et al., 1985).

- Base-ten blocks are a good physical representation of whole numbers and place value, but prompt increased cognitive complexity when representing **decimal numbers.** The difference is both a hindrance and an opportunity, as the designation of the unit block may shift as necessary. For example, the base-ten block representation of the number 2.3 will change if the unit block is the flat or the rod (English and Halford, 1995).

- Students' **conceptual misunderstandings of decimals** lead to the adoption of rote rules and computational procedures that often are incorrect. This adoption occurs despite a natural connection of decimals to whole number, both in notation and computational procedures (English and Halford, 1995).

- The place-value connections (or analogs) between whole numbers and **decimal numbers** are useful for learning, but children often focus directly on the whole number aspects and fail to adjust for the decimal aspects (Hiebert, 1992). For example, a common error is a student's ordering of decimal numbers as if they were whole numbers, claiming 0.56 is greater than 0.7 because 56 is greater than 7. The reading of decimal numbers seemingly as whole numbers (e.g., "point five six" or "point fifty-six") contributes to the previous error (Wearne and Hiebert, 1988b; J. Sowder, 1988).

- Students with a weak understanding of place value have a difficult time **understanding decimals.** For example, students will mentally separate a decimal into its whole number part and its pure decimal part, such as rounding 148.26 to 150.3 (Threadgill-Sowder, 1984). Or, students will assume that "more digits" implies that a number is larger, such as 0.1814 being larger than 0.385 and 0.3 (Hiebert and Wearne, 1986).

- To construct a good **understanding of decimals,** students need to focus on connecting the familiar (e.g., written symbols, place value principles, procedural rules for whole number computations and ordering) with the unfamiliar (e.g., decimal notation and the new quantities they represent). Concrete representations of both the symbols and potential actions on these symbols can help make these connections (Hiebert, 1992).

- Students who connect the **physical representations of decimals** with decimal notation are more apt to create their own procedures for new tasks, such as ordering decimals or converting a decimal to its fractional notation (Wearne and Hiebert, 1988a).

- When students construct an understanding of the **concept of a fraction,** the area model (i.e., a continuous attribute) is preferred over the set model (i.e., a discrete attribute) because the total area is a more flexible, visible attribute. Furthermore, the area model allows students to encode almost any fraction whereas the set model (e.g., group of colored chips) has distinct limitations, especially for a part/whole interpretation. For example, try to represent 3/5 using either four cookies or a sheet of paper (English and Halford, 1995; Hope and Owens, 1987).

- Many teachers have a **surface level understanding of fractions and decimals,** with the result being that students are engaged in learning activities and discussions that are misleading and prompt misconceptions such as "multiplication makes bigger" and "division makes smaller" (Behr et al., 1992).

- Students tend to view **fractions** as isolated digits, treating the numerator and denominator as separate entities that can be operated on independently. The result is an inconsistent knowledge and the adoption of rote algorithms involving these separate digits, usually incorrectly (Behr et al., 1984; Mack, 1990).

- Unlike the situation of whole numbers, a major source of difficulty for students learning fractional concepts is the fact that a **fraction** can have multiple meanings—part/whole, decimals, ratios, quotients, or measures (Kieren, 1988; Ohlsson, 1988).

- Student **understandings of fractions** are very rote, limited, and dependent on the representational form. First, students have greater difficulty associating a proper fraction with a point of a number line than associating a proper fraction with a part-whole model where the unit was either a geometric region or a discrete set. Second, students able to associate a proper fraction on a number line of length one often are not successful when the number line had length two (i.e., they ignore the scaling and treat the available length as the assumed unit) (Novillis, 1976). Finally, though able to form equivalents for a fraction, students often do not associate the fractions 1/3 and 2/6 with the same point on a number line (Novillis, 1980).

- As students build some meaning for the **symbolic representations of fractions,** they overgeneralize their understanding of symbolic representations of whole numbers to fractions and the reverse as well (Mack, 1995).

- Students need to work first with the **verbal form of fractions** (e.g., two-thirds) before they work with the numerical form (e.g., 2/3), as students' informal language skills can enhance their understanding of fractions. For example, the word "two-thirds" can be associated with the visual of "two" of the "one-thirds" of an object (Payne, 1976).

- Students with good understandings of the part/whole interpretation of a **fraction** still can have difficulty with the concept of fraction equivalence, confuse quantity notions with proportionality, possess limited views of fractions as numbers, and have cognitive difficulty relating fractions to division (Kerslake, 1986).

- Students taught the **common denominator method for comparing two fractions** tend to ignore it and focus on rules associated with ordering whole numbers. Students who correctly compare numerators if the denominators are equal often compare denominators if the numerators are equal (Behr et al., 1984).

- Students' **difficulties with ratios** are often due to the different referents involved in the ratio situation (Hart, 1984). A ratio can refer to a comparison between two parts (e.g., 1 can of frozen concentrate to 3 cans of water), a comparison between a part and a whole (e.g., 1 can of frozen concentrate to 4 cans of lemonade), or a comparison between two wholes (e.g., 1 dollar to 4 hours of work).

- Students do not make good use of their understandings of rational numbers as a starting point for developing an **understanding of ratio and proportion** (Heller et al., 1990).

- The unit rate method is clearly the most commonly used and perhaps the best method for working with problems involving **ratios and proportions.** The distinction as "most common" disappears once students are taught then apply by rote the cross-product algorithm for proportions (Post et al., 1985, 1988). Nonetheless, the unit rate method is strongly suggested as "scaffolding" for building proportional reasoning.

- The **cross-product algorithm for evaluating a proportion** is (1) an extremely efficient algorithm but rote and without meaning, (2) usually misunderstood, (3) rarely generated by students independently, and (4) often used as a "means of avoiding proportional reasoning rather than facilitating it" (Cramer and Post, 1993; Post et al., 1988; Hart, 1984; Lesh et al., 1988).

- Students see their work with **ratios** as an additive operation, often replacing the necessary multiplicative concepts with repeated additions (K. Hart, 1981c).

- Students' intuitive understanding of the **concept of infinity** remains quite stable over the middle grades and is relatively unaffected by mathematics instruction (Fischbein et al., 1979).

Computation

- Students learning **multiplication as a conceptual operation** need exposure to a variety of models (e.g., rectangular array, area). Access only to "multiplication as repeated addition" models and the term "times" leads to basic misunderstandings of multiplication that complicate future extensions of multiplication to decimals and fractions (Bell et al., 1989; English and Halford, 1995).

- **Division situations** can be interpreted as either a partition model (i.e., the number of groups is known and the number of members in a group needs to be found) or a measurement model (i.e., the number of members in a group is known and the number of groups needs to be found). Measurement problems are easier for students to model concretely (Brown, 1992), yet partition problems occur more naturally and more frequently in a student's daily experiences. The partition model also is more representative of the long division algorithm and some fraction division techniques (English and Halford, 1995).

- Students learning the processes of **addition and subtraction** need a "rich problem solving and problem-posing environment" that should include:

1. Experiences with addition and subtraction in both in-school and out-of-school situations to gain a broad meaning of the symbols +/-.
2. Experiences both posing and solving a broad range of problems.
3. Experiences using their contextual meaning of +/- to solve and interpret arithmetic problems without a context.
4. Experiences using solution procedures that they conceptually understand and can explain (Fuson, 1992a).

- When performing **arithmetic operations,** students who make mistakes "do not make them at random, but rather operate in terms of meaning systems that they hold at a given time." The teachers feedback should not focus on the student as being "wrong," but rather identify the student's misunderstandings which are displayed "rationally and consistently" (Nesher, 1986).

- **Whole-number computational algorithms** have negative effects on the development of number sense and numerical reasoning (Kamii, 1994).

- Confronted with **decimal computations** such as 4.5+0.26=?, students can respond using either a syntactic rule (e.g., line up the decimal points, then add vertically) or semantic analysis (e.g., using an understanding of place values, you need only add the five-tenths to the two-tenths). The first option relies on a student's ability to recall the proper rules while the second option requires more cognitive understanding on the student's part. Research offers several insights relevant to this situation. First, students who recall rules experience the destructive interference of many instructional and context factors. Second, when confronted with problems of this nature, most students tend to focus on recalling syntactic rules and rarely use semantic analysis. And third, the syntactic rules help students be successful on test items of the same type but do not transfer well to slightly different or novel problems. However, students using semantic analysis can be successful in both situations (Hiebert and Wearne, 1985; 1988).

- The **standard computational algorithms for whole numbers** are "harmful" for two reasons. First, the algorithms encourage students to abandon their own operational thinking. Second, the algorithms "unteach" place value, which has a subsequent negative impact on the students' number sense (Kamii and Dominick, 1998).

- Students view the **multiplication and division algorithms** primarily as "rules to be followed," leading to a persistence that the numbers involved are to be viewed as separate digits and not grouped amounts involving place-values. The result often is an incorrect answer, impacted unfortunately by students' restricted access to their understanding of estimation, place value, and reasonableness of results (Behr et al., 1983; Fischbein et al., 1985; Lampert, 1992).

- Many students never master the **standard long-division algoriths.** Even less gain a reasonable understanding of either the algorithm or the answers it produces. A major reason underlying this difficulty is the fact that the standard algorithm (as usually taught) asks students to ignore place value understandings (Silver et al., 1993).

- Students have great difficulty "admitting" that the answer to a **division** of one whole number by another could contain a decimal or a fraction. The cognitive difficulty is compounded if the task involves division of a number by a number larger than itself. The difficulty seems to reflect a dependence on the partition model for division and a preference for using remainders (M. Brown, 1981a, 1981b).

- Any approach to performing division, including the **long-division algorithm,** requires reasonable skills with proportional reasoning, which in turn requires a significant adjustment in a student's understanding of numbers and the role of using numbers in counting (Lampert, 1992).

- Students constructing meanings underlying an operation such as **long division** need to focus on understanding why each move in an algorithm is appropriate rather than on which moves to make and in which sequence. Also, teachers should encourage students to invent their own personal procedures for the operations but expect them to explain why their inventions are legitimate (Lampert, 1992).

- In a "classic" research study, Silver et al. (1993) showed that when students work with **division problems involving remainders,** their performance is impacted adversely by the students' dissociation of sense making from the solution of the problem. A second important factor is the students' inability to write reasonable accounts of their mathematical thinking and reasoning while solving the division problems.

- Students' use of base-ten blocks improves their understanding of place-value, their accuracy while computing **multi-digit addition and subtraction** problems, and their verbal explanations of the trading/regrouping involved in these problems (Fuson, 1986; Fuson and Briars, 1990). Furthermore, a positive relationship exists between the amount of student verbalizations about their actions while using the base-ten blocks and the students' level of understanding (Resnick and Omanson, 1987).

- Students need a good understanding of the **concept of both a fraction and fraction equivalence** before being introduced to computation situations and procedures involving fractions (Mack, 1993; Bezuk and Bieck, 1993).

- Students learning **computational algorithms involving fractions** have difficulty connecting their concrete actions with manipulatives with their symbolic procedures. Often, a student's personal competence with a rote procedure "outstrips" his/her conceptual understanding of fractions; the unfortunate result is that students cannot monitor their work, can check their answers only by repeating the rote procedure, and are unable to judge the reasonableness of their answer (Wearne and Hiebert, 1988b).

- **Computational algorithms involving fractions** prevent students from even trying to reason or make sense of fraction situations. In fact, students tend to not only remember incorrect algorithms but also have more faith in them compared to their own reasoning (Mack, 1990).

- The **traditional "invert-and-multiply" algorithm for dividing fractions** does not develop naturally from students using manipulatives (Borko et al., 1992). In contrast, the common denominator approach to dividing fractions can be modeled by students using manipulatives and capitalizes on their understanding of the measurement model of whole number division using repeated subtraction (Sharp, 1998).

- Students openly not confident when using fractions **operate with fractions** by adapting or misapplying the computational rules for whole numbers (K. Hart, 1981b).

- Many students solve problems involving **proportions** by using additive strategies which produce incorrect results, not realizing that such problems involve a multiplicative structure (Hart, 1988).

- Students gain little value from being taught the **cross-multiplication algorithm for evaluating a proportion** because of its lack of a conceptual basis (K. Hart, 1981c).

- The **cross-multiplication algorithm for a proportion** is (1) an extremely efficient algorithm but is rote and without meaning, (2) usually misunderstood by students, (3) rarely generated by students independently, and (4) often used as a "means of avoiding proportional reasoning rather than facilitating it" (Cramer and Post, 1993; Post et al., 1988; Hart, 1984; Lesh et al., 1988).

- Students begin with useful **percent strategies** (e.g., using benchmarks, pictorial representations, ratios, and fractions) that are quickly discarded and replaced by their extensive use of school-taught equation strategies. Students' successes with the earlier conceptual strategies have little impact (Lembke and Reys, 1994).

- Students bring **informal and self-constructed computational techniques** into algebra classrooms where more formal methods are developed. Teachers must (1) recognize students who use such informal methods for a given problem, (2) recognize and value these informal methods, and (3) discuss possible limitations of the informal methods (Booth, 1988).

- Young students allowed to develop, use, and discuss **personally invented algorithms** demonstrate enhanced number sense and operational sense (Kamii et al., 1993; J. Sowder, 1992a). These students also develop efficient reasoning strategies, better communication skills, and richer experiences with a wider range of problem solving strategies (Carroll and Porter, 1997).

- The number line is not a good representational model for working with **integer operations,** except for addition. A discrete model (e.g., where the positive elements can cancel the negative elements) is preferred because it has documented success with students and it is more consistent with the actions involved (Kuchemann, 1981a).

- Students tend to avoid using parentheses when doing arithmetic or algebra, believing that the written sequence of the operations determines the **order of computations.** Some students even think that changing the order of the computations will not change the value of the original expression (Kieran, 1979; Booth, 1988).

- Students tend not to view commutativity and associativity as distinct **properties of a number system** (numbers and operators), but rather as “permissions” to combine numbers in any order (Resnick, 1992).

Estimation

- Students need to recognize the **difference between estimation and approximation** in order to select and use the appropriate tool in a computational or measurement situation. Estimation is an educated guess subject to “ballpark” error constraints while approximation is an attempt to procedurally determine the actual value within small error constraints (J. Sowder, 1992a).

- Good estimators tend to have strong self-concepts relative to mathematics, attribute their **success in estimation** to their ability rather than mere effort, and believe that estimation is an important tool. In contrast, poor estimators tend to have a weak self-concept relative to mathematics, attribute the success of others to effort, and believe that estimation is neither important nor useful (J. Sowder, 1989).

- The **inability to use estimation skills** is a direct consequence of student focus on mechanical manipulations of numbers, ignoring operational meaning, number sense, or concept of quantity/magnitude (Reys, 1984).

- The ability to multiply and divide by powers of ten is “fundamental” to the development and use of **estimation skills** (Rubenstein, 1985).

- Three **estimation processes are used by “good” estimators** in Grades 7 through adult. First, **reformulation** massages the numbers into a more mentally-friendly form using related skills such as rounding, truncating, and compatible numbers (e.g., using 6+8+4 to estimate 632+879+453 or using 7200| 60 to estimate 7431| 58). Second, **translation** alters the mathematical structure into an easier form (e.g., using the multiplication 4x80 to estimate the sum 78+82+77+79). And third, **compensation** involves adjustments made either before or after a mental calculation to bring the estimate closer to the exact answer. In this study, the less-skilled students “felt bound” to make estimates using the rounding techniques they had been taught even if the result was not optimal for use in a subsequent calculation (e.g., use of compatible numbers) (Reys et al., 1982).

- Student improvement in **computational estimation** depends on several skills and conceptual understandings. Students need to be flexible in their thinking and have a good understanding of place value, basic facts, operation properties, and number comparisons. In contrast, students who do not improve as estimators seem “tied” to the mental replication of their pencil-and-paper algorithms and fail to see any purpose for doing estimation, often equating it to guessing (Reys et al., 1982; Rubenstein, 1985; J. Sowder, 1992b). Also, good estimators tended to be self-confident, tolerant of errors, and flexible while using a variety of strategies (Reys et al., 1982).

- Teacher emphasis on place value concepts, decomposing and recomposing numbers, the invention of appropriate algorithms, and other rational number sense skills have a long-term impact on middle school students' abilities using **computational estimation.** Rather than learning new concepts, the students seemed to be reorganizing their number understandings and creating new ways of using their existing knowledge as "intuitive notions of number were called to the surface and new connections were formed" (Markovits and Sowder, 1994).

- Students prefer the use of informal **mental computational strategies** over formal written algorithms and are also more proficient and consistent in their use (Carraher and Schliemann, 1985).

- Students' acquisition of **mental computation and estimation skills** enhances the related development of number sense; the key seems to be the intervening focus on the search for computational shortcuts based on number properties (J. Sowder, 1988).

- Experiences with **mental computation** improve students' understanding of number and flexibility as they work with numbers. The instructional key was students' discussions of potential strategies rather than the presentation and practice of rules (Markovits and Sowder, 1988).

- **Mental computation** becomes efficient when it involves algorithms different from the standard algorithms done using pencil and paper. Also, mental computational strategies are quite personal, being dependent on a student's creativity, flexibility, and understanding of number concepts and properties. For example, consider the skills and thinking involved in computing the sum 74+29 by mentally representing the problem as 70+(29+1)+3 = 103 (J. Sowder, 1988).

- The "heart" of flexible **mental computation** is the ability to decompose and recompose numbers (Resnick, 1989).

- The use of a context enhances students' ability to estimate in two ways. First, a **context for an estimation** helps students overcome difficulties in conceptualizing the operations needed in that context (e.g., the need to multiply by a number less than one producing a "smaller" answer). Second, a context for an estimation helps students bypass an algorithmic response (e.g., being able to truncate digits after a decimal point as being basically insignificant when using decimal numbers) (Morgan, 1988).

- Young students tend to use **good estimation strategies** on addition problems slightly above their ability level. When given more difficult problems in addition, students get discouraged and resort to wild guessing (Dowker, 1989).

- Students have a difficult time accepting either **the use of more than one estimation strategy** or more than one estimation result as being appropriate, perhaps because of an emphasis on the "one right answer" in mathematics classrooms. These difficulties lessened as the students progressed from the elementary grades into the middle school (Sowder and Wheeler, 1989).

- Students need to be able to **produce reasonable estimates for computations involving decimals or fractions** prior to instruction on the standard computational algorithms (Mack, 1988; Owens, 1987).

- Students **estimating in percent situations** need to use benchmarks such as 10 percent, 25 percent, 33 percent, 50 percent, 75 percent, and 100 percent, especially if they can associate a pictorial image. Also, student success seems to depend on a flexible understanding of equivalent representations of percents as decimals or fractions (Lembke and Reys, 1994).

RESEARCH ON MEASUREMENT

Attributes and Dimensions

- The research is inconclusive as to the prerequisite relationship between **conservation and a child's ability to measure** attributes. One exception is that conservation seems to be a prerequisite for understanding the inverse relationship between the size of a unit and the number of units involved in a measurement situation. For example, the number expressing the length of an object in centimeters will be greater than its length in inches because an inch is greater than a centimeter (Hiebert, 1981).

- Young children lack a basic understanding of the **unit of measure concept.** They often are unable to recognize that a unit may be broken into parts and not appear as a whole unit (e.g., using two pencils as the unit) (Gal'perin and Georgiev, 1969).

- Figueras and Waldegg (1984) investigated the understanding of **measurement concepts and techniques** of middle school students, with these conclusions:

    1. In increasing order of difficulty: Conservation of area, conservation of length, and conservation of volume.
    2. Measurement units are used incorrectly by more than half of the students.
    3. Students are extremely mechanical in their use of measuring tools and counting iterations of equal intervals.
    4. Students find areas/volumes by counting visual units rather than using past "formula" experiences, even if the counting process is tedious or complex.
    5. Student performance on measurement tasks decreases significantly when the numbers involved are fractions.

    The researchers suggested that "a fixed measuring system is introduced far too early in the curriculum of elementary school, thus creating a barrier to the complete understanding of the unit concept" (p. 99).

- When trying to **understand initial measurement concepts,** students need extensive experiences with several fundamental ideas prior to introduction to the use of rulers and measurement formulas:

    1. **Number assignment:** Students need to understand that the measurement process is the assignment of a number to an attribute of an object (e.g., the length of an object is a number of inches).
    2. **Comparison:** Students need to compare objects on the basis of a designated attribute without using numbers (e.g., given two pencils, which is longer?).
    3. **Use of a unit and iteration:** Students need to understand and use the designation of a special unit which is assigned the number "one," then used in an iterative process to assign numbers to other objects (e.g., if length of a pencil is five paper clips, then the unit is a paper clip and five paper clips can be laid end-to end to cover the pencil).
    4. **Additivity property:** Students need to understand that the measurement of the "join" of two objects is "mirrored" by the sum of the two numbers assigned to each object (e.g., two pencils of length 3 inches and 4 inches, respectively, laid end to end will have a length of 3+4=7 inches) (Osborne, 1980).

- First, the manipulative tools used to help teach number concepts and operations are "inexorably intertwined" with the ideas of measurement. Second, the improved understanding of **measurement concepts** is positively correlated with improvement in computational skills (Babcock, 1978; Taloumis, 1979).

- Students are fluent with some of the simple **measurement concepts and skills** they will encounter outside of the classroom (e.g., recognizing common units of measure, making linear measurements), but have great difficulty with other measurement concepts and skills (e.g., perimeter, area, and volume) (Carpenter et al., 1981).

- Students at all grade levels have great difficulties working with the concepts of **area and perimeter,** often making the unwarranted claim that equal areas of two figures imply that they also have equal perimeters. Perhaps related to this difficulty, many secondary students tend to think that the length, the area, and the volume of a figure or an object will change when the figure or object is moved to another location (K. Hart, 1981a).

- Students initially develop and then depend on physical techniques for **determining volumes of objects** that can lead to errors in other situations. For example, students often calculate the volume of a box by counting the number of cubes involved. When this approach is used on a picture of a box, students tend to count only the cubes that are visible. The counting strategy also fails them if the dimensions of the box are fractions (K. Hart, 1981a).

- The **vocabulary associated with measurement activities** is difficult because the terms are either entirely new (e.g., perimeter, area, inch) or may have totally different meanings in an everyday context (e.g., volume, yard). Furthermore, students do not engage in enough physical measurement activities for the necessary vocabulary to become part of their working vocabulary (K. Hart, 1981a).

Approximation and Precision

- Students need to recognize the **difference between estimation and approximation** before they can select and use the appropriate tool in a computation or measurement situation. Estimation is an educated guess subject to "ballpark" error constraints while approximation is an attempt to procedurally determine the actual value within small error constraints (J. Sowder, 1992a).

- Few researchers have studied the **development of approximation skills** in students, even though approximation is an important tool when mathematics is used in real-world situations. Nonetheless, it is known both that approximation has its own unique skills or rules and that students are unable to use or understand measures of levels of accuracy of an approximation (J. Sowder, 1992a).

- Students experience many **difficulties trying to estimate measurements** of an object if they are unable to use the correct tools to actually measure the object (Corle, 1960).

- After investigating the **measurement estimation abilities** of both students and adults, Swan and Jones (1980) reached these conclusions:

  1. Measurement estimation abilities improve with age.
  2. No gender differences are evident in the estimation of weight or temperature, though males are better estimators of distance and length.
  3. Across all age levels, the best estimates are made in temperature situations and the most difficult estimates involve acreage situations.
  4. Students and adults are poor estimators in measurement situations.

- More than 90 percent of the teaching population agrees that **estimation in measurement situations** is an important skill, but few students experience estimation activities (e.g., less than one lesson on measurement estimation is taught for every 64 lessons on other mathematics) (Osborne, 1980). Facility with measurement estimation skills requires "constant and frequent practice or they will evaporate" (Stake and Easley, 1978).

Systems and Tools

- Hildreth (1979) examined the **measurement strategies used by "good estimators,"** then suggested that students need to learn and practice these strategies:

  1. **Simple comparison:** Ask students to think of a "known" object that is both familiar to them and about the same size as the new object.
  2. **Bracketing:** Ask students to think of two "known" objects such that when they are compared to the new object, one is just slightly smaller and the other is slightly bigger.
  3. **Chunking:** Ask students to partition the new object into parts (not necessarily equal) where they know the measure of these parts.
  4. **Unitizing:** Ask students to create a unit that can be mentally reproduced to form a partition of the new object.
  5. **Rearrangement:** Ask students to mentally cut and rearrange an object to make an estimation easier (especially for area situations).
  6. **Error reduction:** Ask students to identify and discuss systematic errors that can occur in an estimation strategy, then create techniques for compensating for these errors.

RESEARCH ON GEOMETRIC SENSE

Shape and Dimension

- Young students can **define shapes** such as a rectangle or a triangle, but then not use their definitions when asked to point out examples of those shapes. The latter activity is guided by the students' mental prototypes of the shapes, which may differ from their definitions (Wilson, 1986).

- Young students discriminate some **characteristics of different shapes,** often viewing these shapes conceptually in terms of the paths and the motions used to construct the shapes (Clements and Battista, 1992).

- A computer environment can generate **multiple representations of a shape** that help students generalize their conceptual image of that shape in any size or orientation (Shelton, 1985).

- Student **misconceptions in geometry** lead to a "depressing picture" of their geometric understanding (Clements and Battista, 1992). Some examples are:

  1. An angle must have one horizontal ray.

2. A right angle is an angle that points to the right.
3. A segment must be vertical if it is the side of a figure.
4. A segment is not a diagonal if it is vertical or horizontal.
5. A square is not a square if the base is not horizontal.
6. Every shape with four sides is a square.
7. A figure can be a triangle only if it is equilateral.
8. The angle sum of a quadrilateral is the same as its area.
9. The area of a quadrilateral can be obtained by transforming it into a rectangle with the same perimeter.

These conceptual misconceptions often can be traced to a student's focus on a limited number of exemplars of the shape plus the student's tendency to "consider inessential but common features as essential to the concept" (Vinner and Hershkowitz, 1980; Fisher, 1978).

- Students have a difficult time communicating visual information, especially if the task is to communicate a **3-D environment** (e.g., a building made from small blocks) via 2-D tools (e.g., paper and pencil) or the reverse (Ben-Chaim et al., 1989).

- Both teachers and students use an imprecise language that directly impacts the students' developmental progress in **geometric understanding.** In turn, teachers must help students distinguish between the mathematical use of a term and its common interpretation (e.g., plane). Finally, the geometric meaning underlying a student's geometric language may differ considerable from what a mathematics teacher might think is the student's meaning (Clements and Battista, 1992).

- Students have a difficult time using the word **"similar"** and its mathematical meaning correctly. Too often, the word is used loosely by students and teachers to mean "approximately the same," which led to a subsequent classification of rectangles of most dimensions as being similar (K. Hart, 1981c).

Relationships/Transformations

- The van Hieles, after years of extensive research, contend that a student **develops an understanding of geometry** by progressing through five distinct levels in a hierarchical manner similar to those associated with Piaget (Carpenter, 1980; Clements and Battista, 1992):

  1. **Level I—Recognition and Visualization:** Students can name and perceive geometric figures (e.g., squares, triangles) in a global sense and not by their properties. That is, students at this level can recognize and reproduce basic geometric shapes but are unable to identify specific attributes of a shape (e.g., squares have sides that are equal in length) or relationships between shapes (e.g., a square is a rectangle).
  2. **Level II—Analysis:** Students can identify and isolate specific attributes of a figure (e.g., equal side lengths in a square) but only through empirical tests such as measuring. They are unable to make the leap that one geometric property

necessitates associated geometric properties (e.g., the connection between parallelism and angle relationships in a parallelogram).

3. **Level III—Order:** Students understand the role of a definition and recognize that specific properties follow from others (e.g., the relationships between parallelism and angle relationships in a parallelogram) but have minimal skills in using deduction to establish these relationships.
4. **Level IV—Deduction:** Students are able to work within a deduction system—postulates, theorems, and proofs—on the level modeled in Euclid's Elements. (Note: This is the level of the traditional high school geometry course.)
5. **Level V—Rigor:** Students understand both rigors in proofs and abstract geometric systems such as non-Euclidean geometries, where concrete representations of the geometries are not accessible.

The van Hieles suggests further that geometric concepts implicitly understood at one level become explicitly understood at the next level, with a different language operating at each level. Other researchers have made subsequent adjustments in the number and interpretations of the levels in the van Hieles' model.

- Students **investigating relationships in geometry** often experience a "braking effect" similar to that of a mind set. That is, an object can play several roles that may be unrelated, such as a line in a diagram serving both as a transversal between two parallel lines and as the bisector of an angle. Once students notice one role of an object, they have a difficult time noticing the other role(s). Hence, the first role (or concept) has a "braking effect" by masking the second role. Students need to discuss the potential roles an object can assume within a variety of settings if they are to overcome this form of a mind set (Zykova, 1969).

- Students can perform successfully on various assessments in a geometry class, yet hold several **false beliefs.** Examples of these false beliefs are (1) that "geometric form" is preferred over "geometric substance," (2) that a geometry problem not solved in a few minutes is unsolvable, and (3) that geometry (or mathematics) is a collection of facts established by others that "are inaccessible to them except by memorizing" (Schoenfeld, 1988).

- Students build **false interpretations of geometric terms** from their exposure to a limited number of static pictures in texts. For example, many students claim that two lines cannot be parallel unless they are the same length or are oriented vertically or horizontally (Kerslake, 1981).

- Students have some informal understanding of **geometric transformations** such as reflection and rotation, but have a difficult time operating on shapes using these transformations (Kuchemann, 1981b).

- Some students think that the sides of a triangle change length when the triangle is **rotated in a plane** (Kidder, 1976).

- Students working with plane motions in **transformation geometry microworlds** tend to think only in terms of transformations of the figures available and not the transformation of the entire plane (Thompson, 1985).

RESEARCH ON PROBABILITY AND STATISTICS

Chance

- Students of all ages have a difficult time **understanding and using randomness,** with "no marked differences" in understanding within this wide age range. Teachers need to give students multiple and diverse experiences with situations involving randomness and help them understand that randomness "implies that a particular instance of a phenomenon is unpredictable but there is a pattern in many repetitions of the same phenomenon" (Green, 1987; Dessert, 1995).

- Students tend to interpret probability questions as "requests for single **outcome predictions."** The cause of this **probability misconception** is their tendency to rely on causal preconceptions or personal beliefs (e.g., believing that their favorite digit will occur on a rolled die more frequently despite the confirmation of equal probabilities either experimentally or theoretically) (Konold, 1983).

- Students estimating the probability of an event often ignore the **implications of the sample size.** This error is related to an operational misunderstanding of the law of large numbers (Kahneman and Tversky, 1972).

- Students have poor understandings of **fundamental notions in probability:** the use of tree diagrams, spinners using the area model, random vs. nonrandom distributions of objects, and the general idea of randomness itself. To overcome these understandings, students need more exposure to the ratio concept, the common language of probability (e.g., "at least," "certain," and "impossible"), and broad, systematic experiences with probability throughout their education (Green, 1983, 1988).

- Appropriate instruction can help students overcome their **probability misconceptions.** Given an experiment, students need to first guess the outcome, perform the experiment many times to gather data, then use this data to confront their original guesses. A final step is the building of a theoretical model consistent with the experimental data (Shaughnessy, 1977; DelMas and Bart, 1987).

- Students' growth in **understanding probability situations** depends on three abilities that can be developed. First, they need to overcome the "sample space misconception" (e.g., the ability to list events in a sample space yet not recognize that each of these events can occur). Second, they need to apply both part-part and part-whole reasoning (e.g., given four red chips and two green chips, "part-part" involves comparing the two green chips to the four red chips while "part-whole" involves comparing the two green chips to the six total chips). And third, they need to participate in a shared adoption of student-invented language to describe probabilities (e.g., "one-out-of-three" vs. "one-third"). (Jones et al., 1999a).

- Students construct an **understanding of probability concepts** best in learning situations that (1) involve repeatable processes and a finite set of symmetric outcomes (e.g., rolling a die), (2) involve outcomes produced by a process involving pure chance (e.g., drawing a colored chip from a bag of well-mixed chips), and (3) are well recognized as being "unpredictable and capricious" (e.g., predicting the weather). When probability situations deviate from these three prototypes, students experience great difficulty and revert to inappropriate reasoning (Nisbett et al., 1983). These claims appear valid if students are asked to determine the most likely outcomes but are not valid if students are asked to determine the least likely outcomes, a discrepancy due to students' misunderstanding the concept of independence (Konold et al., 1993).

Data Analysis

- Students can calculate the **average of a data set** correctly, either by hand or with a calculator, and still **not** understand when the average (or other statistical tools) is a reasonable way to summarize the data (Gal., 1995).

- Computer environments help students overcome **statistical misconceptions** by allowing them to control variables as they watch a sampling process or manipulate histograms (Rubin and Rosebery, 1990).

- Introducing students prematurely to the **algorithm for averaging data** can have a negative impact on their understanding of averaging as a concept. It is very difficult to "pull" students back from the simplistic "add-then-divide" algorithm to view an average as a representative measure for describing and comparing data sets. Key developmental steps toward understanding an average conceptually are seeing an average as reasonable, an average as a midpoint, and an average as a balance point (Mokros and Russell, 1995).

- Students apply number properties improperly to statistical computations. A primary example is a student who "averages averages" by the "add-them-up-and-divide" algorithm without taking into account the **concept of an appropriate weighting** for each average (Mevarech, 1983).

- Students and adults hold several **statistical misconceptions** that researchers have shown to be quite common:

  1. They assign significance incorrectly to any difference in the means between two groups.
  2. They believe inappropriately that variability does not exist in the real world.
  3. They place too much confidence (unwarranted) in results based on small samples.
  4. They do not place enough confidence in small differences in results based on large samples.
  5. They think incorrectly that the choice of a sample size is independent of the size of the actual population (Landewehr, 19889).

Prediction and Inference

- Six concepts are fundamental to a young child trying to reason in a probability context. These six **probability concepts** are sample space, experimental probability of an event, theoretical probability of an event, probability comparisons, conditional probability, and independence (Jones et al., 1999b).

- As students progress through the elementary grades into the middle grades, their **reasoning in probability situations** develops through four levels:

  1. **Subjective or nonquantitative reasoning:** They are unable to list all of the outcomes in a sample space and focus subjectively on what is likely to happen rather than what could happen.
  2. **Transitional stage between subjective reasoning and naïve quantitative reasoning:** They can list all of the outcomes in a sample space but make questionable connections between the sample space and the respective probability of an event.
  3. **Naïve quantitative reasoning:** They can systematically generate outcomes and sample spaces for one- and two-stage experiments and appear to use quantitative reasoning in determining probabilities and conditional probabilities, but they do not always express these probabilities using conventional numerical notation.
  4. **Numerical reasoning:** They can systematically generate outcomes and numerical probabilities in experimental and theoretical experiments, plus can work with the concepts of conditional probability and independence (Jones et al., 1999b).

- Students often compute the probabilities of events correctly but then use incorrect reasoning when making an **inference about an uncertain event.** The problem is the students' reliance on false intuitions about probability situations that overpower their mathematical computations (Garfield and Ahlgren, 1988; Shaughnessy, 1992).

- Students will focus incorrectly on the single events making up the series when given **probability information about a series of events.** For example, told that that there is a predicted 70 percent chance of rain for ten days, students will claim that it should rain on every one of the ten days because of the high 70 percent value. The underlying problem is known as "outcome orientation" and is prompted by an intuitive yet misleading model of probability (Konold, 1989).

- Students need to make **probability predictions** about possible events in diverse situations, then test their predictions experimentally in order to become aware of and confront both personal misconceptions and incorrect reasoning. Too often, students will discredit experimental evidence that contradicts their predictions rather than restructure their thinking to accommodate the contradictory evidence (DelMas, Garfield, and Chance, 1997).

- Students tend to categorize **events** as equally likely because of their mere listing in the sample space. An example is the student who claims that the probability of rolling a prime number is the same as the probability of rolling a composite number on a single role of a single die (Lecoutre, 1992).

- **Student errors when estimating probabilities** often can be traced to the use of two simplifying techniques which are misleading: representativeness and availability. Using the technique of representativeness, students estimate an event's probability based on the similarity of the event to the population from which it is drawn (e.g., students see the coin flip sequence [HHHHHTHHHH] then claim that H is more probable on the next toss in order to even out the overall probabilities). Using the technique of availability, students estimate an event's probability based on the "ease" with which examples of that event can be produced or remembered (e.g., students' estimations of the probability of rain on a fall day in Seattle will differ if they have recently experienced several rainy days). A residual of the two techniques are these specific errors in making probability prediction:

  1. Disregard for the population proportions when making a prediction.
  2. Insensitivity to the effects of sample size on the ability to make accurate predictions.
  3. Unwarranted confidence in a prediction based on incorrect information.
  4. Fundamental misconceptions of chance, such as the gambler's fallacy.
  5. Misconceptions about the "speed" with which chance data regress to a mean (Shaughnessy, 1981).

- Fischbein and Schnarch (1997) confirmed Shaughnessy's conclusions and added these additional **probability misconceptions:**

  1. The representativeness misconception decreases with age.
  2. The misleading effects of negative recency (e.g., after seeing HHH, feeling the next flip will be a T) decreases with age.
  3. The confusion of simple and compound events (e.g., probability of rolling two 6's equals probability of rolling a 5 and a 6) was frequent and remained.
  4. The conjunction fallacy (e.g., confusing the probability of an event with the probability of the intersection of that event with another) was strong through the middle grades then decreased.

5. The misleading effects of sample size (e.g., comparing probability of two heads out of three tosses vs. probability of 200 heads out of 300 tosses) increased with age.
6. The availability misconception increased with age.

- Students often will assign a higher probability to the **conjunction of two events** than to either of the two events individually. This conjunction fallacy occurs even if students have had course experiences with probability. For example, students rate the probability of "being 55 **and** having a heart attack" as more likely than the probability of either "being 55" or "having a heart attack." An explanation for the error is that students may confuse the conjunction form (e.g., "being 55 **and** having a heart attack") with the conditional form (e.g., "had a heart attack **given that** they are over 55") (Kahneman and Tversky, 1983).

- Students have **difficulties with conditional probabilities** Prob(A | B), attributed to three types of errors:

  1. The Falk Phenomenon (Falk, 1983, 1988) arises when the "conditioning event" occurs after the event that it conditions (e.g., If two balls are drawn without replacement from an urn [WWBB], what is the probability that the first ball was white given that the second ball was white?).
  2. Confusion can arise when trying to identify the correct "conditioning event."
  3. Confusion, especially when diagnosing diseases (Eddy, 1982), between a conditional statement and its inverse (e.g., "the probability that it is raining given that it is cloudy" versus "the probability that it is cloudy given that it is raining").

  Student experiences with real world examples of probability situations will help overcome these misconceptions (Shaughnessy, 1992).

- Student **misconceptions of independent events** in probability situations can be impacted by exposure to real-world experiences that help the students:

  1. Realize that dependence does not imply causality (e.g., oxygen does not cause life yet life depends on oxygen to keep breathing).
  2. Realize that it is possible for mutually exclusive events to not be complementary events.
  3. Realize the distinction between contrary events and contradictory events (Kelly and Zwiers, 1988.

RESEARCH ON ALGEBRAIC SENSE

Relations and Representations

- Schoenfeld and Arcavi (1988) argue that "understanding the concept [of a variable] provides the basis for the transition from arithmetic to algebra and is necessary for the meaningful use of all advanced mathematics." Yet, the **concept of a variable** is "more sophisticated" than teachers expect and it frequently becomes a barrier to a student's understanding of algebraic ideas (Leitzel, 1989). For example, some students have a difficult time shifting from a superficial use of "a" to represent apples to a mnemonic use of "a" to stand for the number of apples (Wagner and Kieren, 1989).

- Students treat **variables** or letters as symbolic replacements for specific unique numbers. As a result, students expect that x and y cannot both be 2 in the equation x+y=4 or that the expression "x+y+z" could never have the same value as the expression "x+p+z" (Booth, 1988).

- Students have difficulty representing and solving algebraic word problems because they rely on a direct syntax approach which involves a "phrase-by-phrase" **translation of the problem into a variable equation** (Chaiklin, 1989; Hinsley et al., 1977). An example of this difficulty is the common reversal error associated with the famous "Students-and-Professors" problem: *Write an equation using the variables S and P to represent the following statement: "There are 6 times as many students as professors at this university." Use S for the number of students and P for the number of professors.* A significant number of adults and students (especially engineering freshmen at MIT) write the reversal "6S=P" instead of the correct expression "S=6P." Clement et al. (1981) suggest that the reversal error is prompted by the literal translation of symbols to words, where S is read as "students" and P as "professors" rather than S as "the number of students" and P as "the number of professors." Under this interpretation, the phrase "6 students are equal to 1 professor" becomes a ratio.

- Students often can describe a procedure verbally yet not be able to recognize the **algebraic representation** of this same procedure (Booth, 1984).

- Students try to force **algebraic expressions** into equalities by adding "=0" when asked to simplify or evaluate (Wagner et al., 1984; Kieren, 1983).

- The **concept of a function** is the "single most important" concept in mathematics education at all grade levels (Harel and Dubinsky, 1992).

- Students have trouble with **the language of functions** (e.g., image, domain, range, pre-image, one-to-one, onto) which subsequently impacts their abilities to work with graphical representations of functions (Markovits et al., 1988).

- Students tend to think every **function** is linear because of its early predominance in most algebra curricula (Markovits et al., 1988). The implication is that nonlinear functions need to be integrated throughout the students' experience with algebra.

- Students, surrounded initially by **function prototypes** that are quite regular, have cognitive difficulties accepting the constant function, disconnected graphs, or piece-wise defined functions as actually being functions (Markovits et al., 1988).

- In Dreyfus' (1990) summary of the research on students' working to **understanding functions,** three problem areas are identified:

  1. The mental concept that guides a student when working with a function in a problem tends to differ from both the student's personal definition of a function and the mathematical definition of a function.
  2. Students have trouble graphically visualizing attributes of a function and interpreting information represented by a graph.
  3. Most students are unable to overcome viewing a function as a procedural rule, with few able to reach the level of working with it as a mathematical object.

- Students' transition into algebra can be made less difficult if their elementary curriculum includes experiences with **algebraic reasoning problems** that stress representation, balance, variable, proportionality, function, and inductive/deductive reasoning (Greenes and Findell, 1999).

- Students may be able to solve traditional problems using both **algebraic and graphical representations**, yet have minimal understanding of the relationships between the two representations (Dreyfus and Eisenberg, 1987; DuFour et al., 1987).

- Graphing technologies encourage students to experiment with mathematics, sometimes with negative effects. In an algebra or precalculus context, visual illusions can arise that actually are student misinterpretations of what they see in a **function's graphical representation.** For example, students view vertical shifts as horizontal shifts when comparing linear graphs (such as the graphs of $y=2x+3$ and $y=2x+5$). Also, students falsely conclude that all parabolas are not similar due to the misleading effects of scaling. Students often conclude that a function's domain is bounded due to misinterpretations of the graphing window (Goldenberg, 1988).

- Students have more facility working with functions represented graphically than functions represented algebraically. The **graphical representations** seem to visually encapsulate the domain, range, informal rule, and behavior of the function in a manner that the algebraic form cannot (Markovits et al., 1988). In turn, high-ability students prefer using the graphical representation, while low-ability students prefer a tabular representation of the function (Dreyfus and Eisenberg, 1981).

- Students **misinterpret time-distance graphs** because they confuse the graph with the assumed shape of the road. Also, students do not necessarily find it easier to interpret graphs representing real-world contexts when compared to graphs representing "symbolic, decontextualized" equations (Kerslake, 1977).

- Students have a difficult time **interpreting graphs,** especially distance-time graphs. Intuitions seem to override their understandings, prompting students to view the graph as the path of an actual "journey that was up and down hill" (Kerslake, 1981).

- Students have difficulty accepting the fact that there are more points on a **graphed line** than the points they plotted using coordinates. This is known as the continuous vs. discrete misconception. Some students even contend that no points exist on the line between two plotted points, while other students accept only one possible such point, namely the mid-point (Kerslake, 1981).

- Middle school students find **constructing Cartesian graphs** difficult, especially with regard to their choice of a proper scaling, positioning the axes, and understanding the structure involved (Leinhardt et al., 1990).

- Precalculus students' use of graphing calculators improved their understanding of the **connections between a graph and its algebraic representation** (in contrast to students learning the same content without calculators). Also, the calculator-using students tended to view graphs more globally (i.e., with respect to continuity, asymptotic behavior, and direction changes) and showed a better understanding of the underlying construction of graphs, especially the use of scale (Rich, 1990).

- The oversimplified **concept of slope** taught to students in an algebra class can lead to misconceptions when working with the concept of slope as a part of differentiation in a calculus class (Orton, 1983).

Operations

- Students experience many difficulties if they persist in viewing algebra as "generalized arithmetic." Some pertinent **algebraic misconceptions or inconsistencies** identified by research studies are:

    1. Arithmetic and algebra use the same symbols and signs but interpret them differently. For example, an equal sign can signify "find the answer" and express an equality between two expressions (Booth, 1988; Matz, 1982).
    2. Arithmetic and algebra use letters differently. For example, students can confuse the expressions 6 m with 6m, where the first represents 6 meters (Booth, 1988).
    3. Arithmetic and algebra treat the juxtaposition of two symbols differently. For example, "8y" denotes a multiplication while "54" denotes the addition 50+4. Another example is the students' inclination that the statement "2x=24" must imply that x=4. (Chalouh and Herscovics, 1988; Matz, 1982).
    4. Students have cognitive difficulty accepting a procedural operation as part of an answer. That is, in arithmetic, closure to the statement "5+4" is a response of "9," while in algebra, the statement "x+4" is a final entity by itself (Booth, 1988; Davis, 1975).

5. In arithmetic word problems, students focus on identifying the operations needed to solve the problem. In algebra word problems, students must focus on representing the problem situation with an expression or equation (Kieran, 1990).

- Students equate learning algebra with learning to **manipulate symbolic expressions** using a set of transformational rules without reference to any meaning of either the expressions or the transformations (English and Halford, 1995).

- **Student errors in using algebra algorithms** often are not due to failing to learn a particular idea but from learning or constructing the wrong idea (Matz, 1980).

- Students experience **difficulty with functions** often because of the different notations. For example, Herscovics (1989) reported that in his research study, 98 percent of the students could evaluate the expression a+7 when a=5 when only 65 percent of this same group could evaluate f(5) when f(a)=a+7.

- Students overgeneralize while **simplifying expressions,** modeling inappropriate arithmetic and algebra analogies. Using the distributive property as the seed, students generate false statements such as a+(bxc)=(a+b)x(a+c), $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$, and $(a+b)^2 = a^2 + b^2$ (Matz, 1982; Wagner and Parker, 1993).

- Elementary students react in different ways when **solving open number sentences** involving multiplication and division (e.g., 6 x _ = 30, 30 ÷ _ = 6, or 30 ÷ 6 = _). First, open number sentences with the operation on the right side of the equality (e.g., 30 = _ x 6) were significantly more difficult than those with the operation on the left. Second, open number sentences with the unknown in the <u>a</u> placeholder position (e.g., _ x 6 = 30) were significantly more difficult for students than when the unknown was in either the <u>b</u> or <u>c</u> placeholder positions. The implication perhaps is either that students use different strategies dependent on the problem format or that certain strategies work better with certain problem formats (Grouws and Good, 1976).

- When **solving equations,** algebra teachers consider the transposing of symbols and performing the same operation on both sides to be equivalent techniques. However, students view the two solution processes as being quite distinct. The technique of performing the same operation leads to more understanding perhaps because it visually emphasizes the symmetry of the mathematical process. Students using the transposition of symbols technique often work without mathematical understanding and are “blindly applying the Change Side-Change Sign rule” (Kieran, 1989).

- Students do not have a good understanding of the **concepts of equivalent equations.** For example, though able to use transformations to solve simple equations (x+2=5 becomes x+2-2=5-2), students seem unaware that each transformation produces an equivalent equation (Steinberg et al., 1990).

- Algebra students lack the kind of "structural conception of algebraic expressions and equations" that is necessary for them to use these **algebraic objects as notational tools** for investigating or proving mathematical relations. Students can formulate correct algebraic generalizations but prefer to confirm the suggested relationships using numerical substitutions. Nonetheless, students do "appreciate" the more general algebraic demonstrations as part of a proof when they are performed by someone else (Lee and Wheeler, 1987).

- An emphasis on the **development of "operation sense"** is necessary to prepare students for their introduction to algebraic reasoning. A suggested approach is the use of word problems and computational problems as contexts for both constructing and enhancing the meanings for the four basic operations (Schifter, 1999).

- Students can learn to interpret the elements of a **matrix** and do matrix multiplication, but their understandings are very mechanical (Ruddock, 1981).

- Students experience difficulties understanding and working with the **concept of a limit.** The underlying problems are (1) the use of common words as mathematical language (e.g., a speed limit is something that cannot be exceeded), (2) the different mathematical interpretations for different contexts (e.g., limit of a sequence, limit of a series, or limit of a function), and (3) the student's false assumption that everything can be reduced to a formula (Davis and Vinner, 1986).

RESEARCH ON PROBLEM SOLVING

- **Interpretations of the term "problem solving"** vary considerably, ranging from the solution of standard word problems in texts to the solution of nonroutine problems. In turn, the interpretation used by an educational researcher directly impacts the research experiment undertaken, the results, the conclusions, and any curricular implications (Fuson, 1992c).

- **Problem posing** is an important component of problem solving and is fundamental to any mathematical activity (Brown and Walter, 1983, 1993).

- Teachers need assistance in **the selection and posing of quality mathematics problems** to students. The primary constraints are the mathematics content, the modes of presentation, the expected modes of interaction, and the potential solutions (concrete and low verbal). Researchers suggest this helpful set of problem-selection criteria:

1. The problem should be mathematically significant.
2. The context of the problem should involve real objects or obvious simulations of real objects.
3. The problem situation should capture the student's interest because of the nature of the problem materials, the problem situation itself, the varied transformations the child can impose on the materials, or because of some combination of these factors.
4. The problem should require and enable the student to make moves, transformations, or modifications with or in the materials.
5. Whenever possible, problems should be chosen that offer opportunities for different levels of solutions.
6. Whatever situation is chosen as the particular vehicle for the problems, it should be possible to create other situations that have the same mathematical structure (i.e., the problem should have many physical embodiments).
7. Finally, students should be convinced that they can solve the problem and should know when they have a solution for it.

Most of these criteria apply or are appropriate to the full grade scale, K–12 (Nelson and Sawada, 1975).

- A **problem needs two attributes** if it is to enhance student understanding of mathematics. First, a problem needs the potential to create a learning environment that encourages students to discuss their thinking about the mathematical structures and underlying computational procedures within the problem's solution. Second, a problem needs the potential to lead student investigations into unknown yet important areas in mathematics (Lampert, 1991).

- Algebra students improve their problem solving performance when they are taught a **Polya-type process for solving problems,** i.e., understanding the problem, devising a plan of attack, generating a solution, and checking the solution (Lee, 1978; Bassler et al., 1975).

- In "conceptually rich" problem situations, the "poor" problem solvers tended to use **general problem solving heuristics** such as working backwards or means-ends analysis, while the "good" problem solvers tended to use "powerful content-**related** processes" (Larkin et al., 1980; Lesh, 1985).

- Mathematics teachers can help students use **problem solving heuristics** effectively by asking them to focus first on the structural features of a problem rather than its surface-level features (English and Halford, 1995; Gholson et al., 1990).

- Teachers' emphasis on specific **problem solving heuristics** (e.g., drawing a diagram, constructing a chart, working backwards) as an integral part of instruction does significantly impact their students' problem solving performance. Students who received such instruction made more effective use of these problem solving behaviors in new situations when compared to students not receiving such instruction (Vos, 1976; Suydam, 1987).

- Explicit **discussions of the use of heuristics** provide the greatest gains in problem solving performance, based on an extensive meta-analysis of 487 research studies on problem solving. However, the benefits of these discussions seems to be deferred until students are in the middle grades, with the greatest effects being realized at the high school level. As to specific heuristics, the most important are the drawing of diagrams, representing a problem situation with manipulative objects, and the translation of word situations to their representative symbolic situations (Hembree, 1992).

- Mathematics teachers who help **students improve as problem solvers** tend to ask frequent questions and use problem resources other than the mathematics textbook. Less successful teachers tend to demonstrate procedures and use problems taken only from the students' textbook (Suydam, 1987).

- Young children often make **errors** when solving mathematical problems because they focus on or are distracted by irrelevant aspects of a problem situation (Stevenson, 1975). This error tendency decreases as students pass through the higher grades, yet the spatial-numerical distracters (i.e., extraneous numbers or diagrams) remain the most troublesome over all grade levels (Bana and Nelson, 1978).

- In their extensive review of research on the problem solving approaches of novices and experts, the National Research Council (1985) concluded that the success of the problem solving process hinges on the **problem solvers' representation of the problem.** Students with less ability tend to represent problems using only the surface features of the problem, while those students with more ability represent problems using the abstracted, deeper-level features of the problem. The recognition of important features within a problem is directly related to the "completeness and coherence" of each problem solver's knowledge organization.

- Young students (Grades 1–3) rely primarily on a **trial-and-error strategy** when faced with a mathematics problem. This tendency decreases as the students enter the higher grades (Grades 6–12). Also, the older students benefit more from their observed "errors" after a "trial" when formulating a better strategy or new "trial" (Lester, 1975).

- While solving mathematical problems, students adapt and extend their existing understandings by both **connecting new information** to their current knowledge and constructing new relationships within their knowledge structure (Silver and Marshall, 1990).

- Students solving a mathematics problem in **small groups** use cognitive behaviors and processes that are essentially similar to those of expert mathematical problem solvers (Artz and Armour-Thomas, 1992).

- **Solution setup** (e.g., organizing data into a table, grouping data into sets, formulating a representative algebraic equation) is the most difficult of the stages in the problem solving process (Kulm and Days, 1979).

- **Problem solving ability** develops slowly over a long period of time, perhaps because the numerous skills and understandings develop at different rates. A key element in the development process is multiple, continuous experiences in solving problems in varying contexts and at different levels of complexity (Kantowski, 1981).

- Results from the Mathematical Problem Solving Project suggest that willingness to take risks, perseverance, and self-confidence are the three most important **influences on a student's problem solving performance** (Webb et al., 1977).

- A reasonable amount of **tension and discomfort** improves the problem solving performance of students, with the subsequent release of the tension after the solution of problem serving as a motivation. If students do not develop tension, the problem is either an exercise or they are "generally unwilling to attack the problem in a serious way" (Bloom and Broder, 1950; McLeod, 1985).

- The elements of **tension and relaxation** are key motivational parts of the dynamics of the problem solving process and help explain why students tend not to "look back" once a problem is solved. Once students perceive "that the problem solution (adequate or inadequate) is complete, relaxation occurs and there is no more energy available to address the problem" (Bloom and Broder, 1950).

- Students tend to speed-read through a problem and immediately begin to manipulate the numbers involved in some fashion (often irrationally). Mathematics teachers need to encourage students to use "**slow-down**" **mechanisms** that can help them concentrate on understanding the problem, its context, and what is being asked (Kantowski, 1981).

- A "vital" part of students' problem solving activity is **metacognition,** which includes both the awareness of their cognitive processes and the regulation of these processes (Lester, 1985).

- Students can solve most one-step problems but have extreme difficulty trying to solve **nonstandard problems,** problems requiring multi-steps, or problems with extraneous information. Teachers must avoid introducing students to techniques that work for one-step problems but do not generalize to multi-step problems, such as the association of "key" words with particular operations (Carpenter et al., 1981).

RESEARCH ON COMMUNICATION

- When teachers increase their **wait times,** the length of the student responses increases, the numbers of student responses increases, the apparent confidence of students in their responses increases, the number of disciplinary interruptions decreases, the number of responses by less able students increases, and students seem to be more reflective in their responses (Rowe, 1978). This study was done in a science classroom, but its results may be applicable to a mathematics classroom as well.

- Students **give meaning to the words and symbols of mathematics** independently, yet that meaning is derived from the way these same words and symbols are used by teachers and students in classroom activities. For communication to occur, the words and symbols must be given meaning in a way that allows teachers "to assess whether the way students understand something fits with his or her understanding or the understanding that is common to the way these symbols [and words] are used in the discipline" (Lampert, 1991).

- Student communication about mathematics can be successful if it involves both the teacher and other students, which may require **negotiation of meanings** of the symbols and words at several levels (Bishop, 1985).

- The teacher has mathematical understandings that allow them to "see" mathematical objects or concepts in ways that learners are not yet ready to "see" themselves. The result is that teachers often "talk past" their students, unless they "see" through their students' understanding (often peculiar) and make the necessary adjustments. Classroom **miscommunication** is well documented by researchers in many areas of mathematics: number sense and place value, basic operations, decimals and fractions, variables, and geometric proofs (Cobb, 1988).

- Teachers need to build an **atmosphere of trust and mutual respect** when turning their classroom into a learning community where students engage in investigations and related discourse about mathematics (Silver et al., 1995). An easy trap is to focus too much on the discourse process itself; teachers must be careful that "mathematics does not get lost in the talk" as the fundamental goal is to promote student learning of mathematics (Silver and Smith, 1996).

- Students **writing in a mathematical context** helps improve their mathematical understanding because it promotes reflection, clarifies their thinking, and provides a product that can initiate group discourse (Rose, 1989). Furthermore, writing about mathematics helps students connect different representations of new ideas in mathematics, which subsequently leads to both a deeper understanding and improved use of these ideas in problem solving situations (Borasi and Rose, 1989; Hiebert and Carpenter, 1992).

- Students **writing regularly in journals** about their learning of mathematics do construct meanings and connections as they "increasingly interpret mathematics in personal terms." The writing sequence that students adopt first is the simple narrative listing of learning events, then progress to personal and more reflective summaries of their mathematics activity, and finally for a few students, create "an internal dialogue where they pose questions and hypotheses" about mathematics. Most students report that the most important thing about their use of journals is "To be able to explain what I think." Also, teachers report that their reading of student journals contributes significantly to "what they knew about their students" and helps them better understand their own teaching of mathematics (Clarke et al., 1992).

RESEARCH ON MATHEMATICAL REASONING

- Summarizing research efforts by the National Research Council, Resnick (1987b) concluded that **reasoning and higher order thinking have these characteristics:**

  1. "Higher order thinking is **nonalgorithmic.** That is, the path of action is not fully specified in advance.
  2. Higher order thinking tends to be **complex.** The total path is not "visible" (mentally speaking) from any vantage point.
  3. Higher order thinking often yields **multiple solutions,** each with costs and benefits, rather than unique solutions.
  4. Higher order thinking involves **nuanced judgment** and interpretation.
  5. Higher order thinking involves the application of **multiple criteria,** which sometimes conflict with one another.
  6. Higher order thinking often involves **uncertainty.** Not everything that bears on the task at hand is known.
  7. Higher order thinking involves **self-regulation** of the thinking process.
  8. Higher order thinking involves **imposing meaning,** finding structure in apparent disorder.
  9. Higher order thinking is **effortful.** There is considerable mental work involved in the kinds of elaborations and judgments required" (p. 3).

- Young children and lower-ability students can learn and use the same **reasoning strategies and higher-order thinking skills** that are used by high-ability students (Resnick et al., 1991).

- **Authentic reasoning in mathematical contexts** can be developed or occur if three conditions are present:

  1. Mathematical reasoning needs to occur in real contexts.
  2. Teachers can motivate mathematical reasoning by encouraging a skeptical stance and by raising key questions about numerical information:
     What information are we still not confident about?
     What relationships do our data fail to reveal?
     What questions did we fail to ask?
     What information did we never monitor?
  3. Mathematical reasoning is both nurtured and natural in collaborative communities Whitin and Whitin (1999).

- Students use **visual thinking and reasoning** to represent and operate on mathematical concepts that do not appear to have a spatial aspect (Lean and Clements, 1981). An example is the visual interpretation and explanations of fraction concepts and operations (Clements and Del Campo, 1989).

- Few high school students are able to **comprehend a mathematical proof** as a mathematician would, namely as a "logically rigorous deduction of conclusions from hypotheses" (Dreyfus, 1990). Part of the problem is that students also do not appreciate the importance of proof in mathematics (Schoenfeld, 1994).

- Teachers can improve their **students' ability to construct and evaluate mathematical proofs** if they (1) transfer to students the responsibility of determining the truth value of mathematical statements and (2) integrate their teaching of proof constructions into other mathematical content rather than treat it as a separate unit (Balacheff, 1988).

- In a study of the **understanding of mathematical proofs** by eleventh grade students, Williams (1980) discovered that:

    1. Less than 30 percent of the students demonstrated any understanding of the role of proof in mathematics.
    2. Over 50 percent of the students stated there was no need to prove a statement that was "intuitively obvious."
    3. Almost 80 percent of the students did not understand the important roles of hypotheses and definitions in a proof.
    4. Less than 20 percent of the students understood the strategy of an indirect proof.
    5. Almost 80 percent of the students did understand the use of a counterexample.
    6. Over 70 percent of the students were unable to distinguish between inductive and deductive reasoning, which included being unaware that inductive evidence does not prove anything.
    7. No gender differences in the understanding of mathematical proofs were evident.

- Students of all ages (including adults) have trouble understanding the **implications of a conditional statement** (e.g., if-then). This trouble is due to a focus on seeking information that verifies or confirms the statement when the focus should be on seeking information that falsifies the statement (Wason and Johnson-Laird, 1972).

- Students' understanding of logical statements is significantly correlated with the frequency of mathematics teachers' use of **conditional reasoning** (e.g., use of "if-then" statements) in their own verbal responses (Gregory and Osborne, 1975).

RESEARCH ON CONNECTIONS

- The call for **making connections in mathematics** is not a new idea, as it has been traced back in mathematics education literature to the 1930s and W.A. Brownell's research on meaning in arithmetic (Hiebert and Carpenter, 1992).

- Though children use different strategies to solve **mathematical problems in out-of-school contexts,** they still develop a good understanding of the mathematical models and concepts they use as tools in their everyday activities (Carraher, Carraher and Schliemann, 1985; Nunes, Schliemann and Carraher, 1993).

- Mathematical meaning plays a vital role in student solutions of problems in everyday activities, especially compared to in-school problem solving activities that depend more on algorithmic rules. The strategies and solutions students construct to **solve problems in real-world contexts** are meaningful and correct, while the mathematical rules used by students in school are devoid of meaning and lead to errors undetected by the student (Schliemann, 1985; Schliemann and Nunes, 1990).

- Students need to build **meaningful connections** between their informal knowledge about mathematics and their use of number symbols, or they may end up building two distinct mathematics systems that are unconnected—one system for the classroom and one system for the real world (Carraher et al., 1987).

- Students need to **discuss and reflect on connections between mathematical ideas,** but this "does not imply that a teacher must have specific connections in mind; the connections can be generated by students" (p. 86). A mathematical connection that is explicitly taught by a teacher may actually not result in being meaningful or promoting understanding but rather be one more "piece of isolated knowledge" from the students' point of view (Hiebert and Carpenter, 1992).

- Learning mathematics in a classroom differs from **learning mathematics outside the school** in these important ways:

    1. Learning and performance in the classroom is primarily individual., while out-of-school activities that involve mathematics are usually group-based.
    2. Student access to tools often is restricted in the classroom, while out-of-school activities allow students full access to tools such as books and calculators.
    3. The majority of the mathematics activities in a classroom have no real-world context or connection, while out-of-school activities do by their very nature.
    4. Classroom learning stresses the value of general knowledge, abstract relationships, and skills with broad applicability, while out-of-school activities require contextual knowledge and concrete skills that are specific to each situation (Resnick, 1987a).

- The **skills and concepts learned in school mathematics** differ significantly from the tasks actually confronted in the real world by either mathematicians or users of mathematics (Lampert, 1990).

- Students learn and master an operation and its associated algorithm (e.g., division), then seem to not associate it with their **everyday experiences** that prompt that operation (Marton and Neuman, 1996).

- Teachers need to choose instructional activities that **integrate everyday uses of mathematics** into the classroom learning process as they improve students' interest and performance in mathematics (Fong et al., 1986).

- Students often can list **real-world applications of mathematical concepts** such as percents, but few are able to explain why these concepts are actually used in those applications (Lembke and Reys, 1994).

- Vocational educators claim that the continual **lack of context in mathematics courses** is one of the primary barriers to students' learning of mathematics (Bailey, 1997; Hoachlander, 1997). Yet, no consistent research evidence exists to support their claim that students learn mathematical skills and concepts better in contextual environments (Bjork and Druckman, 1994).

- Hodgson (1995) demonstrated that the ability on the part of the student to **establish connections within mathematical ideas** could help students solve other mathematical problems. However, the mere establishment of connections does not imply that they will be used while solving new problems. Thus, teachers must give attention to both developing connections and the potential uses of these connections.

Chapter 3

MATHEMATICS IN THE CLASSROOM: WHAT RESEARCH TELLS EDUCATORS

CONSTRUCTIVISM AND ITS USE

- **Constructivism in a nutshell:** Students actively construct **"their** individual mathematical worlds by reorganizing **their** experiences in an attempt to resolve **their** problems" (Cobb, Yackel, and Wood, 1991). The expectation is that the student's reorganized experiences form a personal mathematical structure that is more complex, more powerful, and more abstract than it was prior to the reorganization (Davis et al., 1990).

- Teachers who **implement a constructivist approach** to student learning must try to "see" both their own actions/mathematics and their students' actions/mathematics from their students' perspective (Cobb and Steffe, 1983).

- The **role of teachers and instructional activities** in a constructivist classroom is to provide motivating environments that lead to mathematical problems for students to resolve. However, each student will probably find a different problem in this rich environment because each student has a different knowledge base, different experiences, and different motivations. Thus, a teacher should avoid giving problems that are "ready made" (Yackel et al., 1990).

- A **fundamental principle** underlying the constructivist approach to learning mathematics is that a student's activity and responses are always rational and meaningful to themselves, no matter how bizarre or off-the-wall they may seem to others. One of the teacher's responsibilities is to determine or interpret the student's "rationality" and meaning (Labinowicz, 1985; Yackel et al., 1990).

- **Scaffolding** is a metaphor for the teacher's provision of "just enough" support to help students progress or succeed in each mathematical learning activity. Elaborating on this metaphor, Greenfield (1984) suggests: "The scaffold, as it is known in building construction, has five characteristics: it provides support, it functions as a tool; it extends the range of the worker; it allows the worker to accomplish a task not otherwise possible; and it is used selectively to aid the worker where need be" (p. 118).

- Mathematics teachers must engage in **"close listening" to each student,** which requires a cognitive reorientation on their part that allows them to listen while imagining what the learning experience of the student might be like. Teachers must then act in the best way possible to further develop the mathematical experience of the student, sustain it, and modify it if necessary (Steffe and Wiegel, 1996).

- Young **children enter school with a wide range of self-generated algorithms and problem solving strategies** that represent their a priori conceptual understandings of mathematics. Classroom instruction too often separates the child's conceptual knowledge from the new procedures or knowledge they construct because the "students' informal ways of making meaning are given little attention" (Cobb, Yackel, and Wood, 1991).

- From multiple research efforts on creating a constructivist classroom, Yackel et al. (1990) concluded that "not only are children capable of developing their own methods for completing school mathematics tasks but that **each child has to construct his or her own mathematical knowledge.** That is … mathematical knowledge cannot be given to children. Rather, they develop mathematical concepts as they engage in mathematical activity including trying to make sense of methods and explanations they see and hear from others."

- In her survey of the research on **arithmetic-based learning,** Fuson (1992b) concluded that students can "learn much more than is presented to them now if instruction is consistent with their thinking."

ROLE AND IMPACT OF USING MANIPULATIVES

- In his analysis of 60 studies, Sowell (1989) concluded that "mathematics achievement is increased through the **long-term use of concrete instructional materials** and that students' attitudes toward mathematics are improved when they have instruction with concrete materials provided by teachers knowledgeable about their use" (p. 498).

- Manipulative materials can (1) help students understand mathematical concepts and processes, (2) increase students' flexibility of thinking, (3) be used creatively as tools to solve new mathematical problems, and (4) reduce students' anxiety while doing mathematics. However, several **false assumptions** about the power of manipulatives are often made. First, manipulatives cannot impart mathematical meaning by themselves. Second, mathematics teachers cannot assume that their students make the desired interpretations from the concrete representation to the abstract idea. And third, the interpretation process that connects the manipulative to the mathematics can involve quite complex processing (English and Halford, 1995).

- Students do not discover or understand mathematical concepts simply by manipulating concrete materials. Mathematics teachers need to intervene frequently as part of the instruction process to **help students focus on the**

underlying mathematical ideas and to help build bridges from the students' work with the manipulatives to their corresponding work with mathematical symbols or actions (Walkerdine, 1982; Fuson, 1992a; Stigler and Baranes, 1988).

- Mathematics teachers need much more assistance in both **how to select an appropriate manipulative** for a given mathematical concept and how to help students make the necessary connections between the use of the manipulative and the mathematical concept (Baroody, 1990; Hiebert and Wearne, 1992).

- Manipulatives need to be **selected and used carefully,** as they also can be distracting, ambiguous, and open to misinterpretation. Furthermore, if "multiple embodiments" of a mathematical concept are involved, students might have a hard time focusing on the correct interpretation (Dufour-Janvier, Bednarz and Belanger, 1987).

- In his study of teachers as they used manipulatives in mathematics classrooms, Jackson (1979) documented the existence of these **"mistaken beliefs":**
  1. Almost any manipulative can be used to teach any mathematical concept.
  2. Manipulatives simplify the students' learning of mathematics.
  3. Good mathematics teaching always includes the use of manipulatives.
  4. The number of manipulatives used is positively correlated to the amount of learning that occurs.
  5. Teachers should pick one multipurpose manipulative and use it to teach all or most of the mathematics.
  6. Manipulatives are more useful in the primary grades than in the upper grades.
  7. Manipulatives are more useful with low-ability students than high-ability students.

- Teachers sometimes **overestimate the value of manipulatives** because they as adults are able to "see" the mathematical concepts or processes being represented. Children do not have this "adult eye" (Ball, 1992).

- Concrete materials are **likely to be misused** when a teacher has in mind that students will learn to perform some prescribed activity with them (Boyd, 1992; Resnick and Omanson, 1987; Thompson and Thompson, 1994).

- Teachers need to take into account the **"contextual distance"** between the manipulatives and the mathematical concept to be experienced. For example, base-ten blocks represent place value concepts both physically and physically, while colored chips and money are less visual models because they have no physical features that suggest place value relationships (Hiebert and Cooney, 1992).

- The use of concrete manipulatives do not seem as effective in **promoting algebraic understanding** as they are in promoting student understanding of place value and the basic computational processes (English and Halford, 1995).

- Manipulatives help students at all grade levels **conceptualize geometric shapes and their properties** to the extent those students can create definitions, pose conjectures, and identify general relationships (Fuys et al., 1988).

- In a survey of research on manipulatives, Suydam and Higgins (1977) concluded: “Lessons involving manipulative materials will **produce greater mathematical achievement** than will lessons in which manipulative materials are not used if the manipulative materials are used well” and made these suggestions:

  1. Manipulatives should be used frequently and throughout the instructional program in a manner consistent with the goals of the program.
  2. Manipulatives should be used in conjunction with other learning aids such as diagrams, technology, and resource texts.
  3. Manipulatives should be used by students in a manner consistent with the mathematics content and the mathematics content should be adjusted to maximize the potential of the manipulatives.
  4. Manipulatives should be used with learning activities that are exploratory and deductive in approach.
  5. The manipulatives used should be the simplest possible and yet be the most representative of the mathematical ideas being explored.
  6. Manipulatives should be used with learning activities that include the symbolic recording of results and ideas.

- Manipulatives should be used with **beginning learners,** while older learners may not necessarily benefit from using them (Fennema, 1972).

- Many **secondary students** are at a developmental level that necessitates experiences with both concrete and pictorial representations of mathematical concepts (Driscoll, 1983).

- When students use manipulatives, **achievement is enhanced** across a variety of mathematical topics, at every grade level K–8 and at every ability level (Suydam and Higgins, 1977):

  1. Classroom lessons involving manipulatives have a higher probability of producing greater mathematics achievement than do lessons not using manipulatives.
  2. Students need not necessarily manipulate materials themselves for **all** lessons.
  3. Students do not need to use manipulatives for the same amount of time.
  4. Teacher use of manipulatives decreases from Grade 1 on.

- **Upper primary and middle school students** have considerable difficulties making sense of base-ten blocks without considerable teacher intervention (Labinowicz, 1985).

- Students have “astounding success” using base-ten blocks while **learning the addition and subtraction algorithms** (Fuson and Briars, 1990).

- Students have “consistent success” using concrete materials to motivate their understanding of **decimal fractions and decimal numeration** (Wearne and Hiebert, 1988a).

- Base-ten blocks have little effect on upper-primary students' understanding or use of already **memorized addition and subtraction algorithms** (P. Thompson, 1992; Resnick and Omanson, 1987).

- Students need to **reflect continually on their actions with concrete materials** in relation to the ideas the teacher has worked to establish and in relation to the constraints of the task as they conceive it (Thompson, 1994).

- **Timing is the key.** Once students have learned a rote procedure, it is quite difficult for students to acquire a conceptual understanding of that procedure. Thus, teachers need to focus each student's initial instruction on using manipulatives to build a solid understanding of the concepts and processes involved (Wearne and Hiebert, 1988a).

- Student use of **concrete materials in mathematical contexts** help "both in the initial construction of correct concepts and procedures and in the retention and self-correction of these concepts and procedures through mental imagery" (Fuson, 1992c).

- Students trying to use concrete manipulatives to make sense of their mathematics must first be "**committed to making sense** of their activities and be committed to expressing their sense in meaningful ways" (P. Thompson, 1992).

HOW STUDENTS SOLVE WORD PROBLEMS INVOLVING MATHEMATICS

- Middle school students respond to a word problem with one of **four basic types of strategies** (L. Sowder, 1988):

  1. **Coping Strategies:**
     a. Guess at the operation to be used.
     b. Find the numbers in the problem and add (or subtract, multiply, or divide, depending on recent classroom computational work or the operation the student is most comfortable with).

  2. **Computation-Driven Strategies:**
     a. Look at the numbers in the word problem; they will "tell" you which operation to use (e.g., "if it's like, 78 and maybe 54, then I'd probably either add or multiply. But [78 and 3] looks like a division because of the size of the numbers").
     b. Try all four operations and choose the most reasonable answer.

3. **Slightly Less Immature Strategies:**
    a. Look for “key” words or phrases to tell which operations to use (e.g., “all together” means to add).
    b. Decide whether the answer should be larger or smaller than the given numbers. If larger, try both addition and multiplication and choose the more reasonable answer. If smaller, try both subtraction and division and choose the more reasonable answer.

4. **Desired Strategy:**
    a. Choose the operation which meaning fits the story.

In addition to these strategies, the researchers made three additional conclusions. First, students of all ability levels used immature strategies at times. Gifted students often used an immature, computation-driven strategy (2b). Finally, strategy 3b was used productively by many students, with it failing them only when multipliers less than 1 were involved.

- Teachers can provide several **generic teaching strategies to help students solve word problems,** especially when the student can transfer a known strategy for one problem category to a new problem (Catrambone, 1995, 1996):

    1. Have students discuss carefully chosen exemplary problems and their solutions to minimize gaps in their understandings of the intermediate steps.
    2. Ask students to focus on each problem’s conditions and the effects of the subsequent solution steps.
    3. Help students organize their solution steps into a hierarchically organized goal structure.
    4. Encourage students to label parts of the solution process (as subgoals) as it is implemented.
    5. Challenge students to reflect on and discuss alternate routes to a solution.

- **Drawing diagrams** to represent a problem does facilitate a student’s search for a “relevant” solution for the problem (Larkin and Simon, 1987).

- Students have **difficulty representing and solving algebraic word problems** because they rely on a direct syntax approach which involves a “phrase-by-phrase” translation of the problem into a variable equation (Chaiklin, 1989; Hinsley et al., 1977).

- Fuson et al. (1997) proposed a research-based framework that unifies the mathematics curriculum and the teaching/learning process. It incorporates the **different cognitive phases that students should pass through** when solving word problems (applicable to almost every grade level):

    1. **Building a situation conception:** The students’ focus is on understanding the vocabulary used, understanding the meaning of each sentence, and integrating all of these meanings to understand the problem situation—a good teaching strategy is to ask students to retell the problem in their own words or draw a picture to illustrate the problem.

2. **Forming a mathematical conception:** The students' focus is on the numbers, the unknown(s), and building a representation of the situation using mathematical operations—a good teaching strategy is to ask students to model the situation with concrete manipulatives or diagrams.
3. **Formulating a solution method conception:** The students' focus is on the unknown(s) and the steps needed to determine the values of the unknown(s)—a good teaching strategy is to ask students to discuss their solution plans, the connections they have found in the solution process, and the methods they could use to check the reasonableness of their results.

- Students, confronted with a word problem, experience **cognitive difficulty** if the operation required for the solution procedure is opposite to the operation in the problem's underlying structure (e.g., using subtraction to solve an addition situation). Other factors that create cognitive difficulty are the position of the question in the word problem, the degree of specifics in the wording of the problem, the size of the numbers, the amount of "action cueing" of the operation used in the solution, and the availability of concrete manipulatives (Fuson, 1992c).

- Attempts to determine relationships between **reading ability** and the ability to solve word problems have been inconsistent and inconclusive at best (i.e., varying from no correlation to significant correlations). Educators' assumptions regarding a connection are reasonable but not supported by the research (Lester, 1980; Hembree, 1992). The one consistent factor is reading comprehension skills (Suydam, 1985).

- Students' **spatial ability** often is strongly correlated to their problem solving ability (Moses, 1977; Wilson and Begle, 1972). However, spatial visualization abilities are related to mathematics achievement differently for females than for males (Sherman, 1979; Suydam, 1985).

- **Computational ability** is strongly correlated with the ability to solve word problems for young students; this relationship decreases in the higher grades (Dodson, 1972; Jerman, 1974; Knifong and Holton, 1976; Meyer, 1978). However, good computational skills do not guarantee success in solving problems. For example, in one research study, 76 percent of the 9-year-olds and 96 percent of the 13-year-olds could subtract a two-digit number from a two-digit number, but only 59 percent and 87 percent respectively could solve a word problem based on that same subtraction problem. When the situation involved multiplication of fractions, the success rate dropped from 69 percent to 20 percent for the 13-year-olds (Kantowski, 1981).

- Young students often **try to solve word problems** by first circling all of the numbers, then work backwards from the problem's end, searching for the "key word" that signals the arithmetic operation needed (Schoenfeld, 1985a).

- Algebra teachers should be aware that **word problems that can be solved without algebra** (i.e., with arithmetic) can be counterproductive. Rather, algebra students should be given problems that "discourage the search for arithmetic solutions," a search process on the teacher's part that can be as simple as changing the whole numbers in word problems to decimals (MacGregor and Stacey, 1996).

- Algebra students know that **categorization of word problems by type** (e.g., mixture, distance-time-rate, etc.) provides information that can be useful in the solution of algebra problems (Hinsely et al., 1977):

    1. Students are aware of problem categories, agree on these categories, and can recognize them in problem contexts.
    2. Students usually recognize the category quickly when reading a problem, sometimes as early on as reading the initial noun phrase.
    3. Students have access to useful information (e.g., appropriate equations and diagrams) about the different problem categories that can be used to help solve new problems.
    4. Students often become overly dependent on associating problem categories with specific story content, which then prevents their transfer of solution processes to problems that are solved in the same way but have different story content.

- Based on their research with **student attitudes while solving mathematics problems,** Brown et al. (1989) suggested that "Math word problems … are generally encoded in a syntax and diction that is common only to other math problems…. By participating in such ersatz activities students are likely to misconceive entirely what practitioners actually do. As a result, students can easily be introduced to a formalistic, intimidating view of math that encourages a culture of math phobia rather than one of authentic math activity" (p. 34).

- Students have significantly more **difficulty with word problems** if the numbers involved refer to continuous units of measurement (e.g., 7 inches) rather than discrete units (e.g., seven marbles). Unfortunately, the majority of the standard text problems involve a discrete context (M. Brown, 1981a).

- The **presence of decimals** in a word problem makes it significantly harder for students to determine the correct operations to perform (M. Brown, 1981a).

- Considered one of the "more surprising findings," Nesher et al. (1982) discovered that **word problems involving the same logical structure and the same mathematical operation** "behave differently" from the student's perspective. For example, students were almost twice as successful on the subtraction problem "Dan had 10 dollars. How many dollars are left, if Dan has spent 3?" compared to the subtraction problem "Joseph and Ronald had seven marbles altogether. Three of them were Joseph's. How many of them were Ronald's?" Thus, the factor of semantic category (e.g., combine, change, or compare situations) gains importance along with its mathematical structure.

- Bell et al. (1984, 1989) documented with research what most teachers have known for a long time: **students avoid reading** the text of word problems, trying to solve the problems by focusing only on the numbers in the problems.

MASTERY OF BASIC FACTS AND ALGORITHMS

- The last 60 years of educational research supports the conclusion of Brownell and Chazal (1935) that **drill with a fact or skill** does not guarantee immediate recall. However, student competence with a mathematical skill does necessitate extensive practice (Bjork and Druckman, 1994). The research is conclusive that drill alone contributes little or nothing to growth in a student's mathematical understanding.

- **Meaningful instruction and drill** go together as part of a successful learning experience, but meaningful instruction must precede drill or practice (Dessert, 1981). A balanced approach to both is needed in mathematics classrooms, as students who can access both memorized and meaningful ideas in mathematics achieve at a higher level than those who rely on either one without the other (Askew and William, 1995).

- Synthesizing both his research and other related research, Davis (1978) produced these guiding principles for using **drill of number facts** in a productive manner:

  1. Students should not try to memorize a number fact that they do not understand (e.g., rote recall that 2+3=5 versus being able show why 2 added to 3 is 5).
  2. Students should participate in drill knowing that the goal is to memorize facts that they understand.
  3. Teachers need to emphasize remembering, not explaining, during drill sessions.
  4. Teachers need to keep drill sessions brief yet include them on a daily basis.
  5. Students should focus on only a few facts at a time, while also reviewing previously memorized facts.
  6. Teachers need to express confidence in each student's ability to memorize all of the number facts.
  7. Students should receive immediate feedback.
  8. Teachers need to stress that the two key aspects of facts recall are accuracy and speed.
  9. Teachers need to vary drill activities and be enthusiastic during the activity.
  10. Teachers need to praise students on their progress in memorizing facts, keeping a record of their individual progress (i.e., self-improvement, not competition against others or an artificial barrier).

- **Practice toward mastery of basic skills and procedural algorithms** should not occur until students develop the meaning underlying those skills or algorithms. Research results (and frustrated teachers) consistently suggest that if this practice occurs too soon for a student, it is very difficult for that student to step back and focus on the meaning that should have been developed at the very beginning (Brownell and Chazal., 1935; Resnick and Omanson, 1987; Wearne and Hiebert, 1988a; Hiebert and Carpenter, 1992).

- Students trying to master the basic addition facts should be given experiences with the **derived fact strategies.** For example, 5+6 can be transformed into [5+5]+1, which can be solved by the sum of the easier double [5+5]=10 and 1. Because this strategy builds on a student's number sense and meaningful relationships between basic combinations, it improves fact recall and provides a "fall-back" mechanism for students (Fuson, 1992a; Steinberg, 1985).

- Students have **conceptual trouble with the zero and one multiplication facts** and often resort to rote memory. The difficulty perhaps is due to interference created by a dependence on the repeated-addition model for multiplication. For example, a child easily becomes confused trying to add 5 to itself either one time or zero times (Cooney et al., 1988; Campbell and Graham, 1985).

- Students need to master the basic facts and practice algorithmic procedures. **The more efficiently a procedure is executed,** the more mental effort that is saved to focus on other related tasks. However, such mastery and practice does not seem to contribute to the development of meaning of the underlying mathematics (Hiebert, 1990).

- **Student errors in using algorithms** are usually not caused from failing to learn a particular idea but from learning or constructing the wrong mathematical idea (Brown and Burton, 1978).

- It is not clear that large amounts of practice are necessary or even the best way to promote recall of the more "complex algorithms," such as computations with multi-digit numerals, fractions, or algebraic expressions. These algorithmic procedures are probably remembered and implemented better if additional time is spent making sense of them (Hiebert and Lefevre, 1986; Skemp, 1978). **If teachers want students to remember algorithmic procedures,** teachers need to ask them to step back and think about the procedures" (Hiebert, 1990).

- "To a great extent **children adapt the algorithms they are taught** or replace them by their own methods; it is only when these methods fail them that they see a need for a rule at all. We appear to teach algorithms too soon, illustrate their use with simple examples (which the child knows he can do in another way) and assume once taught they are remembered. We have ample proof that they are not remembered or sometimes remembered in a form that was never taught, e.g., to add two fractions, add the tops and add the bottoms.

  The teaching of algorithms when the child does not understand may be positively harmful in that what the child sees the teacher doing is 'magic' and entirely divorced from problem solving" (K. Hart, 1981d).

USE AND IMPACT OF COMPUTING TECHNOLOGIES

- Based on a survey of the research, Weaver (1981) concluded: " … **when calculators were used in the variety of ways** investigated to date across a rather wide range of grade levels and content areas, evidence suggests that we have **no cause for alarm or concern** about potentially harmful effects associated with calculator use. This is particularly true with respect to computational performance…. Seldom is the research literature so clear as it is in this respect" (p. 158).

- In a recent study of the **long-term effect of young children's use of calculators,** Groves and Stacey (1998) formed these conclusions:

  1. Students will not become reliant on calculator use at "the expense of their ability to use other methods of computation."
  2. Students who learn mathematics using calculators have higher mathematics achievement than noncalculator students—both on questions where they can choose any tool desired and on mental computation problems.
  3. Students who learn mathematics using calculators demonstrate a significantly better understanding of negative numbers, place value in large numbers, and especially decimals.
  4. Students who learn mathematics using calculators perform better at interpreting their answers, especially again with decimals.

- Calculators can successfully help **introduce basic concepts of algebra** through extensive explorations with numerical computations (Demana and Leitzel, 1988).

- In a study of students **using calculators while learning calculus,** Gimmestad (1982) concluded that:

  1. Students sometimes change their solution approaches because of their access to calculators.
  2. Students using calculators are more effective when exploring ideas or solution approaches within a problem context.
  3. Students using calculators are much more likely to check their work by retracing steps.
  4. Students using calculators achieve overall at the same level as students without calculators.

- **Calculators enhance** student's (1) use of deductive reasoning, (2) ability to elaborate retrospectively on their strategies, (3) use of specific problem solving techniques to reach successful solutions, and (4) ability to evaluate their progress while solving problems (Kelly, 1985).

- Calculator-using students outperform noncalculator-using students on **calculator-neutral tests** (where calculator use is optional on the students' part). Great care and rigor must be used in preparing items for such tests to ensure that the intended mathematical objectives are being assessed; it may even be necessary to norm the scores for the two groups (calculator-using vs. noncalculator-using) independently (Harvey, 1992).

- Reviewing research on the **impact of calculator use in the mathematics classroom,** Wheatley (1980) and Shumway et al. (1981) jointly concluded:

  1. Students using calculators experience a far greater number and variety of mathematical concepts and computations.
  2. Students weak in their recall of basic facts become more successful as problem solvers when given access to calculators.
  3. Students using calculators express more confidence when attacking mathematical problems.
  4. Students using calculators exhibit more exploratory behaviors when solving mathematical problems.
  5. Students using calculators spend more time on attacking problems and less time on computing.
  6. Student and teacher attitudes toward mathematics improve when calculator use becomes part of the classroom routine.

- In their survey of 79 research studies, Hembree and Dessart (1986) concluded: “At all grades but Grade 4, a **use of calculators in concert with traditional mathematics instruction** improves the average student’s basic skills with paper and pencil, both in working exercises and in problem solving…. Across all grade and ability levels, students using calculators possess a better attitude toward mathematics and an especially better self-concept in mathematics than students not using calculators” (p. 83).

- Students using **graphing** calculators in a precalculus class tend to perform better on critical thinking measures than those students not using calculators (Farrell, 1989).

- **Gender differences** in performance on graphing tasks tend to disappear in college algebra classes when students use graphing calculators (Dunham, 1990).

- Graphing calculators **change the nature of classroom interactions and the role of the teacher,** prompting more student discussions with the teachers playing the role of consultants (Farrell, 1990; Rich, 1990).

- **Graphing calculators facilitate algebraic learning** in several ways. First, graphical displays under the student’s control provide insights into problem solving (e.g., a properly scaled graph motivates the discovery of data relationships). Second, graphical displays paired with the appropriate questions (e.g., data points, trends) serve as assessments of student reasoning at different levels (Wainer, 1992).

- “The **graphing calculator gives the student the power** to tackle the process of ‘making connections’ at her own pace. It provides a means of concrete imagery that gives the student a control over her learning experience and the pace of that learning process … an active conversation between the student and the calculator for, indeed, the calculator responds to the prompts of the student—it does not act on its own." Furthermore, calculator use helps students see mathematical connections, helps students focus clearly on mathematical concepts, helps teachers teach effectively, and especially supported female students as they become better problem solvers (Shoaf-Grubbs).

- **Computer environments impact student attitudes** and affective responses to instruction in algebra and geometry. In addition to changing the social context associated with traditional instruction, computer access provides a mechanism for students to discover their own errors, thereby removing the need for a teacher as an outside authority (Kaput, 1989).

- Students learn more advanced mathematics in less time and with enhanced conceptual understanding in a **symbolic-manipulative computer environment** (Palmiter, 1986).

- Computer environments can help students **overcome statistical misconceptions** because students control the important variables as they watch a sampling process or manipulate histograms (Rubin and Rosebery, 1990).

- Students need experiences with **computer simulations,** computer spreadsheets, and data analysis programs if they are to improve their understanding of probability and statistics (Shaughnessy, 1992).

- Yerushalmy et al. (1986) investigated the impact of **rich geometric environments created on a computer** (e.g., *The Supposer*) and made these conclusions:

  1. Students of all ability levels can conceptualize the construction of a geometric figure in dynamic terms and accept it as a general prototype.
  2. Students have trouble making conjectures based on their explorations or data.
  3. Students do not think it is necessary to justify their generalizations from a finite number of cases.

- Students learning in a **computer environment and related algebra curriculum** perform better on mathematical modeling tasks, general problem solving tasks, and even tasks involving standard algebra manipulations (J. Fey and K. Heid's algebra project results reported by Kieran, 1990).

- Computer environments such as microworlds and simulations **provide informal assessments** in the form of "snapshots" of student work as they explore mathematical ideas (Shaughnessy, 1992).

- Dynamic geometry software programs create rich environments that **enhance students' communications using mathematics** and help students build connections between different mathematical ideas (Brown et al., 1989).

- Interactive computing technologies **enhance both the teaching and learning of mathematics.** Great benefits occur if the technology's power (1) is controllable by either the students or teachers, (2) is easily accessible in a way that enables student explorations, and (3) promotes student generalizations (Demana and Waits, 1990).

- Graphing calculators produce data tables that help students explore meaningful problem environments prior to the **learning of standard algebraic techniques.** The end result is the students' enhanced understanding of the dynamics of change and the meaning underlying algebraic expressions (Heid and Kunkle, 1988).

- Students using graphing tools on a calculator or a computer have **trouble with scaling when studying functions and their graphs.** A common misconception is that changes in scale also change the values on the graph, with students unaware that the real change is their perception of the graph or the amount of the graph visible on the screen (Leinhardt et al., 1990).

- The power of calculators and computers make "the organization and structure of algebra problematic." Easy access to graphic representations and symbolic manipulators **reduce the need to manipulate algebraic expressions or to solve algebraic equations** (Romberg, 1992).

- Graphing options on calculators provide **dynamic visual representations** that act as "conceptual amplifiers" for students learning algebra. Student performance on traditional algebra tasks is improved, especially relative to the development of related ideas such as transformations or invariance (Lesh, 1987).

- Graphing calculators or software can lead students to **new misunderstandings of graphs.** An example is a student's conclusion that a graph is "jagged," prompted by pixel limitations of the technology (Moschkovich et al., 1993).

CULTURE OF THE MATHEMATICS CLASSROOM

- Multiple studies of the **culture of the mathematics classroom** have concluded that the student's role and actions depend primarily on the view of mathematics "projected" by the teacher. In her summary of the relevant research, Nickson (1992) concluded: "The linearity and formality associated with most teaching of mathematics from published schemes or textbooks tend to produce a passive acceptance of mathematics in the abstract, with little connection being made by pupils between their work and real life. Pupils accept the visibility of mathematics in terms of a 'right or wrong' nature, and their main concerns seem to be with the quantity of mathematics done and its correctness…. When beliefs about mathematics differ and where views of mathematics as socially constructed knowledge prevail, pupils take on quite a different role. The messages they receive are that they are expected to contribute their own ideas, to try their own solutions, and even to challenge the teacher … (p. 110)."

- **Teachers with "an integrated, conceptual understanding" of mathematics** tend to organize their classrooms and learning activities that encourage students to engage and interact with the conceptual aspects of mathematics. Furthermore, the depth of the mathematics taught correlates highly with the depth of the teachers' mathematical knowledge (Fennema and Franke, 1992).

- **Warm and supportive teachers** are more effective than critical teachers (Titunoff et al., 1975; Rosenshine and Furst, 1971).

- Teachers **maintain student engagement at doing mathematics** at a high level if they (1) select appropriate tasks for the student, (2) support proactively the student's activity, (3) ask students consistently to provide meaningful explanations of their work and reasoning, (3) push students consistently to make meaningful connections, and (4) do not reduce the complexity/cognitive demands of the task. Student engagement in mathematical activities declines if teachers (1) remove the challenging aspects of the tasks, (2) shift the students' focus from understanding to either the correctness or completeness of an answer, or (3) do not allow an appropriate amount of time for students to complete the task (Henningsen and Stein, 1997).

- A crucial role of the teacher is to structure "a pervasive norm in the classroom that **helping one's peers to learn** is not a marginal activity, but is a central element of students' roles" (Slavin, 1985).

- In a review of 80 research studies on grouping in mathematics classrooms, Davidson (1985) concluded that **students working in small groups** significantly outscored students working individually in more than 40 percent of the studies. Students working as individuals in a mathematics classroom performed better in only two of the studies (and Davidson suggests that these studies were faulty in design).

- Students working on **solving a mathematics problem in small groups** exhibit cognitive behaviors and processes that are essentially similar to those of expert mathematical problem solvers (Artz and Armour-Thomas, 1992).

- **Learning mathematics in cooperative groups** is effective, especially for younger students. When students reach high school, the research evidence is less clear, as these students exhibit stronger individual motivations, interact socially in more complex ways, and often are defensive or embarrassed about their knowledge and learning in mathematics (Steen, 1999).

- The research conclusions on the effect of **cooperative learning in mathematics classrooms** are quite consistent (Davidson, 1990; Davidson and Kroll, 1991; Leiken and Zaslavasky, 1999; Slavin, 1985; Weissglass, 1990):

    1. Students with different ability levels become more involved in task-related interactions.
    2. Students' attitudes toward school and mathematics become more positive.
    3. Students often improve their problem solving abilities.
    4. Students develop better mathematical understanding.
    5. The effects on students' mathematics achievement have been positive, negative, and neutral.

- Students working in **cooperative groups outperform individuals** competing against each other. A meta-analysis of 800 studies on problem solving suggests that the differing factor is the generation of more problem solving strategies by cooperative groups than by individuals working competitively (Qin et al., 1995).

- Teachers can **maximize mathematical learning in a small group environment** by engaging students in learning activities that promote "questioning, elaboration, explanation, and other verbalizations in which they can express their ideas and through which the group members can give and receive feedback" (Slavin, 1989).

- **Students solving mathematical problems in small groups** invokes three features that enhance the individual student's cognitive (re)organization of mathematics:

    1. The student experiences "challenge and disbelief" on the part of the other members of the group, which forces them to examine their own beliefs and strategies closely.
    2. The group collectively provides background information, skills, and connections that a student may not have or understand.
    3. The student might internalize some of the group's problem solving approaches and make them part of their personal approach (Noddings, 1985).

- Teachers trying to **build and sustain mathematical discourse** amongst students need to create an environment in which students build a "personal relationship" with mathematics. Three key elements need to be in this environment:

    1. Students need to engage in authentic mathematical inquiries.
    2. Students must act like mathematicians as they explore ideas and concepts.
    3. Students need to negotiate the meanings of, and the connections among, these mathematical ideas with other students in the class (D'Ambrosio, 1995).

- Several factors influence or maintain **student engagement** at the level necessary to do quality mathematics. The primary factors are: high-quality tasks that build on students' prior knowledge of mathematics, effective scaffolding on the teachers' part, an appropriate amount of time to engage in the mathematics, both teacher and student modeling of high performance actions, and a sustained effort by the teacher to ask for explanations and meaning (Henningsen and Stein, 1997).

- Several teacher actions help establish **a classroom culture that supports mathematical discourse** (Yackel and Cobb, 1993):

    1. Have a routine of setting norms for both small-group and large-group activities.
    2. Address student and group expectations in class.
    3. Insist that students solve personally challenging problems.
    4. Insist that students explain their personal solutions to peers.

5. Insist that students listen to and try to make sense of the explanations of others.
6. Insist that students try to reach consensus about solutions to a problem.
7. Insist that students resolve any conflicting interpretations or solutions.
8. Capitalize on specific incidents when a student's activity either "instantiated or transgressed a social norm" by rediscussing the classroom expectations.

- Teachers need to do more than ask questions in a mathematics classroom, as the **cognitive level of the questions** being asked is very important. Though the research is quite depressing in regard to teachers' use of questioning, it is quite consistent:

1. 80 percent of the questions asked by mathematics teachers were at a low cognitive level (Suydam, 1985).
2. During each school day, there were about five times as many interactions at low cognitive levels than at high cognitive levels (Hart, 1989).
3. Low cognitive level interactions occurred about 5.3 times more often than high cognitive level interactions (Fennema and Peterson, 1986).
4. In an average of 64.1 interactions in a 50-minute class period, 50.3 were low-level cognitive interactions, 1.0 involved high-level cognitive interactions, and the remaining interactions were not related to mathematics (Koehler, 1986).

- Students tend not to **correct their own errors** because of either an unwillingness or an inability to search for errors. Most students are "just too thankful to have an answer, any answer, to even dare to investigate further" (K. Hart, 1981d).

IMPACT OF ABILITY GROUPING

- In his meta-analysis of more than 70 research studies on ability grouping (K–12), Begle (1975) concluded **bright students benefited more** on both cognitive and affective measures from ability grouping. It seemed to make no difference for other students.

- In his similar meta-analysis 15 years later, Slavin (1987, 1990) reached slightly different conclusions:

1. The **overall effects of ability grouping** on the achievement of elementary students in all content areas is negligible.
2. The overall effects of ability grouping on the achievement of secondary students in all content areas is negligible.
3. The one exception is when students are grouped homogeneously for mathematics/reading instruction on the basis of their mathematics/reading achievement scores. That is, homogeneous grouping for math instruction can have positive effects on a student's mathematics achievement when instructional materials appropriate for that student's level of performance are used.

- Mathematics teachers ask **high-achieving students** to do exactly the same mathematical activities as average-achieving students for 84 percent of their instructional time (Westberg et al., 1992).

- Mathematics teachers act differently when teaching **general mathematics students** than when they teach algebra students. First, they give less amounts of organized instruction (lesson with discussion) to general mathematics students. And second, general mathematics students subsequently spend the majority of their class time doing homework as seatwork, yet receive less assistance and encouragement during this time from the teacher (Confrey and Lanier, 1980).

INDIVIDUAL DIFFERENCES AND EQUITY ISSUES

- Differences in the overall mathematics ability of males and females are "negligible or nonexistent" (Deaux, 1985; Hyde and Linn, 1986). However, the wide publication of **gender differences** and the corresponding lack of publication regarding the lack of gender differences have combined to influence female perceptions of their ability to learn mathematics (Damarin, 1990).

- Research attempts to determine **differences in problem solving ability between males and females** remain inconclusive. Nonetheless, the gender differences that have been observed seem to increase with age, perhaps explainable by affective variables such as confidence in learning mathematics and mathematics as a male domain (Wilson, 1972; Fennema and Sherman, 1978).

- Females tend to achieve higher than males on **lower-level cognitive problems** in mathematics, while males tend to achieve higher than females on more complex cognitive problems (Fennema, 1981).

- Mathematics teachers **ask more questions of male students than female students,** plus they give male students more opportunities to respond to higher-level cognitive questions (Fennema et al., 1980). Mathematics teachers interact more with male students than with female students, especially relative to blame or praise interactions (Becker, 1979).

- Differences in student mathematics achievement based on either their **social classes or racial/ethnic group** can be "detected almost as soon as students can be reliably tested. They are pervasive, evident across all mathematical content domains and skills levels, and increasing over time" (p. 639) (Secada, 1992).

- Most **minority students** end up in a general or remedial mathematics course as their terminal mathematics course at the high school level (Oakes, 1987). Furthermore, academic counselors do not encourage minority students to take advanced courses in mathematics (Dossey et al., 1988; Myers and Milne, 1988; Oakes, 1987).

- **Language minority students** tend to underachieve and are underrepresented in mathematics courses. An important need is the development of these students' understanding and mastery of English as the primary language of instruction (Cuevas, 1984). To help meet this need, mathematics teachers can take these three necessary actions (Cuevas, 1990):

1. Mathematics teachers and students need to discuss language (English and mathematics) in terms of its different skills—listening, reading, writing, and speaking.
2. Mathematics teachers and students need to distinguish between the language of daily communication and mathematics as a language.
3. Mathematics teachers need to both allow for and incorporate the students' native language in the development of mathematics skills and conceptual understanding. For example, research has documented that mathematical skills and concepts can be developed when language minority students are allowed to discuss the mathematics activities in their native language (Saville-Troike, 1984; Cummins, 1984; Hakuta, 1986).

TEACHER ATTITUDES AND STUDENT ATTITUDES

- A **student's attitude toward mathematics** is not a one-dimensional construct. Just as there are different types of mathematics, there potentially are a variety of attitudes towards each type of mathematics (Leder, 1987).

- Despite inconsistent definitions of **attitude relative to mathematics,** Fennema (1977) suggests that research supports these "tentative conclusions":

  1. A significant positive correlation exists between student attitudes and mathematics achievement—this relationship increases as students proceed through the grades.
  2. Student attitudes toward mathematics are quite stable, especially in Grades 7–12.
  3. The middle grades are the most critical time period in the development of student attitudes toward mathematics.
  4. Extremely positive or negative attitudes tend to predict mathematics achievement better than more neutral attitudes.
  5. Gender-related differences in attitudes towards mathematics exist, perhaps related to similar gender-related differences in confidence or anxiety measures relative to learning mathematics.

- Students develop **positive attitudes toward mathematics** when they perceive mathematics as useful and interesting. Similarly, students develop negative attitudes towards mathematics when they do not do well or view mathematics as uninteresting (Callahan, 1971; Selkirk, 1975). Furthermore, high school students' perceptions about the usefulness of mathematics affect their decisions to continue to take elective mathematics courses (Fennema and Sherman, 1978).

- **High levels of positive feeling toward mathematics and intrinsic motivation** are important prerequisites for student creativity, student use of diverse problem solving strategies, and deep understanding of mathematics (McLeod and Adams, 1989; Schiefele and Csikszentmihalyi, 1995).

- Elementary students **think that mathematics is difficult.** In fact, if something is easy, they conclude that it cannot involve mathematics. As they get older and the once difficult mathematics now seems easy, students adjust their view of mathematics to ensure that it is difficult and unfamiliar (Kouba and McDonald, 1987).

- Research on student locus-of-control has revealed a **growing amount of "fatalism"** in middle school mathematics classrooms that seems to persist through high school. Students state that they are "not in control" when they try to solve a mathematics problem, as "something or somebody else pulls all the strings." Furthermore, the attitude can develop into a belief that they cannot solve mathematics problems (Haladyna et al., 1983).

- The **development of positive mathematical attitudes** is linked to the direct involvement of students in activities that involve both quality mathematics and communication with significant others within a clearly defined community such as a classroom (van Oers, 1996).

- One out of every two students thinks that **learning mathematics is primarily memorization** (Kenny and Silver, 1997).

- **Students rapidly lose or ignore meaning in mathematics** when in middle school, a time when symbols take on "a life of their own" with few connections being made to conceptual representations. The result is an overuse of rote rules or procedures with no effort on the students' part to evaluate the reasonableness of any answers they obtain (Wearne and Hiebert, 1988b).

- **Students believe** that mathematics is important, difficult, and based on rules (Brown et al. (1988). Similarly, students associate mathematics with uncertainty, with knowing, and being able to get the right answer quickly (Schoenfeld, 1985b; Stodolsky, 1988). In turn, Lampert (1991) suggests that "these cultural assumptions are shaped by school experience, in which doing mathematics means remembering and applying the correct rule when the teacher asks a question, and mathematical truth is determined when the answer is ratified by the teacher" (p. 124).

- Students exhibit **four basic "dysfunctional" mathematical beliefs** (Borasi, 1990; based on a review of Buerk, 1981, 1985; Oaks, 1987; Schoenfeld, 1985a):

  1. The goal of mathematical activity is to provide the correct answer to given problems, which always are well defined and have predetermined, exact solutions.
  2. The nature of mathematical activity is to recall and apply algorithmic procedures appropriate to the solution of the given problems.
  3. The nature of mathematical knowledge is that everything (facts, concepts, and procedures) is either right or wrong with no allowance for a gray area.

4. The origin of mathematical knowledge is irrelevant—mathematics has always existed as a finished product which students need to absorb as transmitted by teachers.

- **Students' beliefs about mathematics** can weaken their ability to solve nonstandard problems in mathematics. For example, if students believe that they should be able to complete every mathematics problem in five minutes or less, they will not persevere when trying to solve a problem that requires more than five minutes. The problem is that these beliefs are built from their perceptions of mathematics that is experienced continually in classroom situations (Schoenfeld, 1985a; Silver, 1985).

- Teachers confront "critical moments" in their mathematics classroom by making decisions that reflect their **personal beliefs about mathematics** and how it should be taught (Shroyer, 1978).

- **Student attitudes toward mathematics** correlate strongly with their mathematics teacher's clarity (e.g., careful use of vocabulary and discussion of both the why and how during problem solving) and ability to generate a sense of continuity between the mathematics topics in the curriculum (Campbell and Schoen, 1977).

- The **attitude of the mathematics teacher** is a critical ingredient in the building of an environment that promotes problem solving and makes students feel comfortable to talk about their mathematics (Yackel et al., 1990).

- Many students respond intensely and negatively when they confront word problems that involve multiplying or dividing by decimals less than one. This **negative reaction** is triggered by the students' solution process (though possibly correct) because it contradicts their expectations that in mathematics "multiplication makes bigger, division makes smaller" (L. Sowder, 1989).

- Students who attribute their success in mathematics to high ability or effort will be **motivated to learn mathematics.** In contrast, students who attribute their lack of success in mathematics to low ability or the material's difficulty will not be motivated to study mathematics and expect not to be able to learn mathematics. Mathematics teachers need to intervene to help the unmotivated students realize that success in learning mathematics is related to effort (Weiner, 1984).

- Teaching children to both **set personal learning goals and take responsibility** for their own learning of mathematics leads to increased motivation and higher achievement in mathematics (DeCharms, 1984).

- **Teacher feedback to students** is an important factor in a student's learning of mathematics. Students who perceive the teacher's feedback as being "controlling and stressing goals that are external to them" will decrease their intrinsic motivation to learn mathematics. However, students who perceive the teacher's feedback as being "informational" and that it can be used to increase their competence will increase their intrinsic motivation to learn mathematics (Holmes, 1990).

- Mathematics teachers have little understanding of the actual beliefs of students relative to their **intrinsic motivation in mathematics classrooms.** Thus, teachers build mathematics lessons based on their personal conceptions of intrinsic motivation, which may not be appropriate in every situation. When given techniques for both giving attention to and being able to predict student beliefs, mathematics teachers are able to "fine-tune their instruction to turn kids on to mathematics" (Middleton, 1995).

- Mathematics **teachers need to focus their students' motivation and persistence** on both "deriving meaning from the mathematics task rather than just getting the task done" and "developing independent thinking skills and strategies for solving mathematics problems rather than on obtaining the one right answer to the mathematics problem" (p. 12) Peterson (1988).

- **Students "learn" to believe** that mathematical processes are "foreign to their thinking." This learned belief causes them subsequently to forgo common sense and not use the wealth of their personal knowledge when solving mathematical problems (Baroody and Ginsburg, 1990).

- **Students' views of mathematical truth** and the value they place on the practicality of mathematics often interfere with their conceptual growth in mathematics, especially in regard to their appreciation of the need for formal thinking (Williams, 1991).

- **Confidence scores** are good predictors of students' decisions on enrolling in elective mathematics classes, especially for female students (Sherman and Fennema, 1977).

- **High-confidence students** have more interactions about mathematics with their teachers than low-confidence students and these interactions tend to be on a higher cognitive level. Mathematics teachers perhaps are unconsciously sending a message to the low-confidence students that they also have less ability in mathematics and thus should expect less of themselves mathematically (Reyes, 1980).

- A meta-analysis of 26 research studies concludes that there is a consistent, negative correlation between **mathematics anxiety and achievement in mathematics.** This correlation is consistent across all grade levels, both gender groups, and all ethnic groups (Ma, 1999). The most effective ways to reduce mathematics anxiety are a teacher's use of systematic desensitization and relaxation techniques (Hembree, 1990).

USING PERFORMANCE-BASED ASSESSMENT

- **External assessment expectations and instruments** have a "profound influence" on what teachers, administrators, and parents' value in the classroom. Because teachers know the format and characteristics of these assessments, they tend to adjust their teaching and curricula to reflect this knowledge. However, the changes teachers make are not always consistent with professional recommendations such as the *Curriculum and Evaluation Standards in School Mathematics* (NCTM 1989). For example, more than a fourth of the teachers in a research survey reported that they had decreased their emphasis on calculator use despite the suggested reverse action by professional associations (Romberg et al., 1989).

- Almost half of the **administrators responsible for assessment** at the district level hold "beliefs about the alignment of tests with curriculum and teaching that are based on behaviorist learning theory, which requires sequential mastery of constituent skills and behaviorally explicit testing of each learning step" Shepard (1991).

- Research supports the **current trends toward alternative assessments.** Mathematics instruction and mathematics assessments must be "interdependent" to achieve the maximum benefits of performance-based assessments (Frederiksen and Collins, 1989; Linn et al., 1991; Wolf et al., 1991). Furthermore, students need both access to their assessment results and assistance in using these results as learning tools to reflect on their strengths and weaknesses in mathematics (Nitko, 1989).

- Secondary students "learn usable knowledge and skills more effectively and efficiently through experiences" with **open-ended mathematics problems** than with traditional goal-specific problems or exercises. When solving goal-specific mathematics problems, students use strategies that successfully solve the specific problem at hand but are "less effective for making connections among concepts and procedures for organizing knowledge." When solving open-ended mathematics problems, students create, adapt and use solution strategies that "make important relationships more salient. Thereby helping students to develop knowledge that is better organized and skills that are more usable" (Sweller and Levine, 1982; Sweller, Mawer, and Ward, 1983; Owen and Sweller, 1985).

- The **use of open-ended problems** in a mathematics classroom is a teacher's "best chance" to assess a student's level of understanding or development of meaning in the mathematics (Davis, 1978).

- High school mathematics teachers tend to assess students' mathematical understanding using standardized tests and text-associated tests as a narrow basis. More **open-ended assessment tasks need to be used by mathematics teachers** as part of their regular evaluations of students (Senk et al., 1997).

- **Open-ended assessment tasks** require students to communicate in a mathematical context that reveals both the level and the quality of their understanding of mathematics (Magone et al., 1994).

- Several challenges face both students and teachers when **open-ended performance assessments and rubric scoring** are incorporated into a mathematics classroom (Peressini and Webb, 1999):

  1. Students need many opportunities to become familiar with this new form of assessment and to become "comfortable performing on demand."
  2. Students need to realize the importance of communicating their mathematical reasoning in a variety of formats.
  3. Students need to accept and respond appropriately when requested to "leave clear trails" of their computations and other mathematical work.
  4. Teachers need to gain confidence in their ability to correctly analyze student responses in this more complex assessment environment.
  5. Teachers need to be open to unconventional student responses and try to follow these responses "through the eyes of their students."
  6. This new assessment process can be quite time-consuming for both students and teachers.

- Conceptual understanding of the four basic operations incorporates **connections among representations**—concrete, pictorial., symbolic, and real-world. Assessments in mathematics classrooms need to focus on these connections as they have great influence on students' ability to use their conceptual understanding in problem situations (Huinker, 1990).

- A **variety of alternative assessments in mathematics must be used** to generate the information a mathematics teacher needs to determine what his/her students are thinking, how his/her students are reasoning, and what the next instructional steps should be (Thompson and Senk, 1993; Gay and Thomas, 1993).

- Students at all grade levels tend not to use **strategies to self-monitor and self-evaluate** their mathematical work and processes (Schoenfeld, 1985b).

- Research on the **content and construct validity of standardized tests in mathematics** documents that they reveal "very little as to how much mathematics a student knows," despite the traditional assumptions or expectations of teachers, administrators, and parents (Haertel, 1985).

- Based on research with mathematics teachers and the use of **alternative assessments** in classroom environments, Cooney et al. (1993) concluded that "however innovative the tasks, teachers will not use them for assessment if:

  1. These tasks do not reflect their own understanding of mathematics.
  2. They do not recognize the value of such tasks in measuring significant mathematical knowledge.
  3. They do not value the outcomes the items purport to measure" (p. 247).

- From research work with **performance-based assessments** over a decade, Aschbacher (1991) concludes that they must have these key features:

  1. Students need to be asked to produce, do, or create something that requires higher level thinking or problem solving skills.
  2. Students need to respond to assessment tasks that are meaningful, challenging, engaging, and instructional.
  3. Students should face assessment tasks set in real-world contexts or close models.
  4. Students' process behavior must be assessed equally along with a product.
  5. Criteria and standards for performance need to be public knowledge and made known to students in advance.

- **Teachers tend to plan** in terms of classroom learning activities rather than in terms of content performance outcomes (Clark and Yinger, 1979).

Chapter 4

OTHER RESEARCH AND ISSUES

PROFESSIONAL DEVELOPMENT PROGRAMS FOR MATHEMATICS TEACHERS

- In a comprehensive review of the research on **teachers' thought processes and decision making**, Clark and Peterson (1986) formed these broad conclusions:

  1. Thinking plays an important part in teaching.
  2. Teachers' plans have real consequences in the classroom.
  3. Teaching is an interactive process that forces teachers to continually think and make decisions.
  4. Teachers have personal theories and belief systems that impact their classroom perceptions, plans, and actions.
  5. Teachers have beliefs and knowledge that affect how they perceive and think about inservice training, new curricula, and the extent to which they implement the training and curricula originally intended by the developers.

- The degree of **consistency between a mathematics teacher's conceptions and his/her actions** as a teacher depends on the teacher's disposition to reflect on his/her actions—within the context of his/her beliefs about teaching and mathematics, his/her students, the subject matter, and the classroom environment (Thompson, 1984).

- Mathematics teachers often have the same **mathematical misunderstanding and "naïve conceptualizations"** that are demonstrated by their students (Lesh and Schultz, 1983; Post et al., 1985).

- **Mathematics teachers construct their own knowledge** and ways of knowing in a manner similar to their students. The implication is that professional development activities need to encourage teachers to reorganize their pedagogical content knowledge and beliefs by resolving their own problems (e.g., classroom situations) (Cobb, Yackel, and Wood, 1991).

- **One-time inservice workshops** are "unlikely" to produce either significant or long-term change in mathematics teachers—their teaching approaches, their beliefs, their attitudes, or their mathematical understanding. At best, one-time inservice workshops can promote awareness and are a “good kick-off” for more long-term staff development opportunities (Fullan and Steigelbauer, 1991; Little, 1993).

- **“One-shot” inservice workshops** are not adequate for the needs of professional development in mathematics education. To be effective and responsive to the current needs, a professional development program should (Lovitt et al., 1990):

    1. Focus on issues or concerns identified by the mathematics teachers themselves.
    2. Be as close as possible to the mathematics teacher’s classroom environment.
    3. Be extended over a significant period of time.
    4. Be openly supported by both mathematics teachers and their administrators.
    5. Integrate opportunities for mathematics teachers to reflect, discuss, and provide feedback.
    6. Give mathematics teachers a genuine sense of ownership of the activities and desired outcomes.
    7. Expect a conscious commitment on the part of each participating mathematics teacher.
    8. Involve a group of mathematics teachers from a school rather than an individual teacher.
    9. Involve a mathematics education consultant or “critical friend” in the development of workshop activities.

- The **two major influences on a teacher’s professional growth** are access to innovative classroom materials and the opportunity to reflect on classroom events (Clarke, 1997).

- Districts need to build **teacher inservice programs** around quality activities that help mathematics teachers (1) examine their beliefs and practices, (2) develop intrinsic motivations for investigating alternative teaching and assessment methods as part of their practices, and (3) develop personal justifications for their practices as a teacher (Thompson, 1992).

- **An intervention program for mathematics teachers** can impact how teachers teach geometry, what geometry ideas they teach, and their expectations of student learning of geometry as a process (Swafford et al., 1997).

- Teachers can **change their belief systems significantly** but three new views of the teacher’s “self” are required: one’s occupational identity as a teacher, one’s sense of competency as a teacher, and one’s self-concept as a teacher (Fullan and Stiegelbauer, 1991).

- **Staff collegiality** is an important part of the teacher’s professional environment if a mathematics teacher is to change his/her classroom environment. Five elements or behaviors define the desired collegiality (Driscoll, 1986; Little, 1982):

    1. Mathematics teachers need to talk frequently with other mathematics teachers about how mathematics can be taught, learned, and assessed.

2. Mathematics teachers need both the time and access to observe other mathematics teachers (which includes being observed in return).
3. Mathematics teachers need to investigate, plan, adapt, and implement the mathematics curriculum as a group.
4. Mathematics teachers need to teach each other what they know about teaching and learning mathematics.
5. Mathematics teachers need to support each other as they take risks.

- The teaching and learning process has been characterized as lying on a linear continuum, with the extremes being **imposition and negotiation.** In their broad sociological studies, Goodlad (1983) and Stake and Easley (1978) documented that mathematics and science, as taught and learned at the elementary level, tends to hover near the imposition extreme. Evidence of this "camp" are these observations:

  1. Teachers believe that elementary school mathematics is traditional arithmetic, which is comprised of basic skills and computational algorithms.
  2. Teachers treat learning basic facts and skills as instructional goals "isolated" from conceptual meaning or context.
  3. Teachers depend on textbooks as their curriculum guide.
  4. Teachers tend to use direct instruction or demonstration, followed by paper-and-pencil exercises to be done individually.
  5. Teachers respond to student errors and misunderstandings by repeating their original instruction and practice routines.
  6. Teachers view alternative techniques and ideas constructed by students as "undesirable behaviors to be eliminated."

  Note: We can only hope that these results are dated and no longer valid!

- Many research studies identify the effectiveness of certain teacher behaviors or characteristics (often in clusters), but no study has isolated conclusively a teacher behavior that is directly related to increased mathematics achievement for all students and under all conditions. Nonetheless, the research studies concur that **increasing the amount of time devoted to mathematics instruction** (and the time students actively attend to that instruction) does lead to increased achievement (Grouws, 1980).

- **Case studies** are print or video materials that offer narrative accounts of a teaching episode and tend to raise a "teaching dilemma." When case studies are included as part of professional development programs, mathematics teachers tend to confront their own mathematical understandings, reflect on their own students' thinking in mathematics, and often try out new ideas (e.g., curricular approaches, questioning strategies) in their classrooms (Davenport and Sassi, 1995; Schifter, 1994).

- When **teachers teach unfamiliar topics,** they tend to talk longer, tend to rely on low-level cognitive questions, tend to use seat work during most of the class time, and tend to avoid using laboratory (or hands-on) projects (Carlsen, 1990). Though this research project focused on how science is taught in classrooms, the conclusions perhaps extend to mathematics classrooms as well.

- Professional development programs that help teachers make significant changes can lead to a **growing isolation of the teachers** involved. In one sense, this isolation is a purposeful strategy on the teachers' part for conserving energy, given their workday is spent attending to multiple needs and varied demands of students with limited support. Educational reform can occur only if teachers receive increased levels of support and/or a reduction in some of their daily responsibilities (Flinders, 1988).

- Research consistently documents that four **teacher characteristics or actions** are critical collectively to the support of effective instruction (Ball, 1990, 1993; Brown and Borko, 1992; Leinhardt and Smith, 1985; Post et al., 1991; Shulman, 1987; Thompson, 1992; Cobb et al., 1991; Little, 1993; Loucks-Horsley, 1994; Mahr, 1988; Shifter and Simon, 1992):

  1. Teachers need deep understandings of mathematics they teach—concepts, practices, principles, representations, and applications.
  2. Teachers need a deep understanding of the ways that children learn mathematics.
  3. Teachers need to implement pedagogies that elicit and build upon students' thinking about mathematics.
  4. Teachers need to engage continually in analytic reflection on their practice.

- To provide some perspective, these **attributes or behaviors of an effective mathematics teacher** were identified prior to the current reform movement in mathematics education (Good and Grouws, 1977; Evertson et al., 1980):

  1. Exhibits general clarity of instruction.
  2. Creates a task-focused environment.
  3. Is nonevaluative (i.e., little praise or criticism).
  4. Creates a relaxed learning environment.
  5. Demonstrates higher achievement expectations.
  6. Has relatively few behavior problems.
  7. Teaches class as a unit.
  8. Demonstrates alternative approaches for responding to problems.
  9. Emphasizes the meaning of mathematical concepts.
  10. Spends more time on content presentations and discussions than on seat work.
  11. Builds systematic review procedures into their instructional plans.
  12. Has more efficient transitions and student attentiveness.

- **Preservice teacher education programs** produce many new teachers who (1) lack a sufficient knowledge or deep understanding of mathematics, (2) are unable to build quality lessons that focus on mathematical meaning, and (3) are unable to interpret their students' thinking about mathematics (Fuson, 1992c).

CHANGES IN HOW TEACHERS TEACH MATHEMATICS AND HOW STUDENTS LEARN MATHEMATICS

- **Students in reform-oriented mathematics curricula** (compared to traditional programs) perform better in assessments of mathematics understanding of concepts but poorer on assessments of computational ability (Dessert, 1981).

- **Mathematics content decisions at the secondary level** are impacted primarily by seven different factors (Cooney, Davis, and Henderson, 1975):

  1. Requirements or regulations from governing bodies (e.g., OSPI or the Legislature).
  2. Objectives developed by a teacher, a department, or a district committee.
  3. The students' expected use of the content to be taught.
  4. The students' interest shown in learning the content.
  5. The teachers' interest in teaching the content.
  6. The predicted difficulty of the content.
  7. Authoritative standards expressed by professional groups (e.g., National Council of Teachers of Mathematics or Washington State Mathematics Council).

- Teachers' **content-decisions differ and are impacted** by their mathematical knowledge, their interest and enjoyment in teaching mathematics, their beliefs in the importance of mathematics, and their expectations of what students can achieve (Porter et al., 1988). A distinct minority of teachers make content decisions based on their strong convictions about mathematics (and these teachers often are not the ones with the greatest mathematical knowledge) (Freeman, 1986).

- **Teachers' prior experiences as students** learning mathematics in school settings have a strong impact on their subsequent practice and beliefs as professional teachers of mathematics. In this regard, Ball (1987) described the need of teachers to "unlearn to teach mathematics."

- The **responsibilities of the teacher as a professional** have been redefined by the reform movement: A mathematics teacher today is responsible for understanding how each student constructs a personal understanding of mathematics within the complex environment of the ongoing mathematics classroom (Steffe, 1988).

- Three elements are **basic requirements if positive reform is to occur** in how mathematics as it is both taught and learned (Lovitt et al., 1990):

  1. Mathematics teachers must reflect on their current practices and then be encouraged to develop, in very practical terms, a clear vision of what the suggested changes in mathematics education imply for their own personal behavior and role as a mathematics teacher.
  2. Mathematics teachers need access to exemplary curriculum materials that help them reflect on their current roles as teachers, try out new roles, and modify their actions as teachers in line with the "accumulated experience" of the many teachers involved in the development and testing of the materials.

3. Mathematics teachers need access to a motivating and well-structured inservice program that focuses on supporting their professional growth as they try to reshape how students learn mathematics in their classrooms.

- "The goal of many research and implementation efforts in mathematics education has been to promote learning with understanding. But achieving this goal has been like searching for the Holy Grail. There is a persistent belief in the merits of the goal, but **designing school learning environments** that successfully promote understanding has been difficult" (p. 65) (Hiebert and Carpenter, 1992).

- The words "slow and difficult to achieve" best describe the **classroom changes suggested by professional guidelines** for improving mathematics education (Cooney, 1985; 1987). The primary hindrances are teachers' beliefs regarding the nature of mathematics (e.g., as a formal., external structure of knowledge rather than a human activity). These beliefs subsequently impact the teachers' view on how mathematics needs to be taught (even though they often do not believe that this is the best way to teach mathematics) (Dossey, 1992).

- Clarke (1997) identified 12 **factors that influence teachers as they try to change** their role and actions in mathematics classrooms (listed in random order):

  1. The educational reform movement in general.
  2. The principal and school community supporting the teacher.
  3. Access to internal support personnel in the teacher's building.
  4. A spirit of collegiality, collaboration, and experimentation on a teaching staff.
  5. The building of grade-level teams of teachers.
  6. Access to innovative curriculum materials.
  7. Access to an extended and varied inservice program.
  8. The availability and input of external support personnel.
  9. Access to an educational researcher as an audience and a critical friend.
  10. Establishment of outcomes (goals and assessments) valued by the teacher.
  11. The day-to-day conditions under which the teacher works.
  12. The teacher's knowledge or understanding of mathematics.

NEXT STEPS: USING RESEARCH AS EDUCATORS

Step 1: Mathematics Teachers Accepting Responsibility for Change

The research is definitive on this point. Though a legislature, district, or professional group may "demand" that changes occur in how mathematics is taught and learned in the mathematics classroom, the teacher is the primary participant and decision-maker in the change process. To make the necessary changes into a reality, mathematics teachers must be both reflective and proactive professionals. Passivity and appeals to tradition are not acceptable if mathematics teachers are to be responsible for aligning their curriculum and instruction with the EALRs and the WASL. Both are current realities, not options.

Where does a mathematics teacher start? Again, research is helpful. Attitudes favorable to change and a deeper knowledge of mathematics are certainly two important factors. A complementary and broad knowledge of research results in mathematics education can help mathematics teachers be reflective and proactive. In turn, teachers' beliefs and understanding of mathematics can be impacted, supported, and changed by knowledge of research results.

Research in mathematics education has other uses that support teachers trying to change. For example, innovative problem tasks used in research projects can be adapted for use in classrooms as part of instruction or assessment. Also, methodological approaches used in research projects can be adapted for classroom use, e.g., "think-aloud" probes or questions can be adapted to enhance teacher-student interactions.

Step 2: The Reeducation of Mathematics Teachers

The majority of the mathematics teachers in Washington State learned mathematics in a system or environment before the EALRs and the WASL were developed. Unfortunately or fortunately, the necessary reeducation of mathematics teachers must focus on both mathematics understanding and pedagogy, a lengthy process that requires a long-term commitment on the part of the Legislature, each district, each building, each administrator, and each teacher.

Again, we suggest that searching through the research is a relevant activity and part of this reeducation process. For example, while searching for research results relevant to the task of this text, we discovered a multitude of interesting suggestions and activities for teaching mathematics within the framework defined by the EALRs and the WASL. This document would become more valuable if these suggestions and activities were included, but that task was beyond the goal of this text. Another resource text perhaps needs to be produced. Or, we suggest that you travel on a similar journey through the research literature; you will be amazed at the great ideas lying hidden in the growing mountain of mathematics resource texts. Good starting places are James Hiebert's (1999) discussion of research relative to the foundation and implementation of the NCTM Standards and Mike Battista's (1999) discussion on how ignorance of research results leads to incorrect teacher decisions.

Step 3: Mathematics Teachers in Their New Roles as Researchers

Each day in the classroom, mathematics teachers are learning—about how they teach mathematics, how students learn mathematics, how to assess mathematical learning, and how to use mathematical resources. This learning process is informal research on the teacher's part, devoid of the stilted language and statistical graphics that tend to "shroud" attitudes toward research. One step toward reflective change is to make this role of the teacher as a researcher more concrete.

Districts could reasonably engage their mathematics teachers in modified versions of "action research," a process first introduced in the 1940s. In current interpretations of informal action research, the classroom teacher and his/her administrator become problem solvers; their overall task is to define and investigate a question that is directly linked to the mathematical learning of their students. Once the relevant research is examined, teachers and administrators (working alone or in teams) do informal research, gather data, discuss their findings with their colleagues, and make decisions based on their project's outcomes. The implications of informal action research within a classroom or district can be powerful, as documented by several research studies. Wood (1988) concluded that educators directly involved in informal research tend to use other research results more often. And, Holly (1991) concluded that "action research as a major form of professional development, is now seen as central to the restructuring of schools" (p. 133).

A second reasonable approach is for mathematics teachers to become critical users of research. Suydam and Weaver (1975) ask mathematics teachers to "remember that just because research says something was best for a **group** of teachers in a **variety** of classrooms, doesn't necessarily mean that it would be best for **you** as an individual teacher in your **particular** classroom…. Teachers have individual differences as well as pupils!… Teachers must be careful not to let prior judgments influence their willingness to try out and explore: open-mindedness is important…. Be willing to investigate" (p. 6). In turn, we hope that this resource text has provided some ideas for you to explore or investigate in your mathematics classrooms.

A subsequent step that a few teachers might take is that of collaboration with someone who specializes in mathematics education research. The collaboration must be on equal terms—in motivation, in decision making, and in responsibility. Research in mathematics education should be bidirectional from the classroom teacher's point of view. In one direction, classroom teachers should try to be knowledgeable about the available research results and the ways they can be integrated into their classrooms. In the other direction, the classroom teacher is a primary source for identifying concerns, problems, or questions that need to be addressed by educational researchers. If the two groups can communicate, a powerful collaborative effort is created that can significantly impact the mathematics teacher's classroom specifically and the mathematics education community in general (Silver, 1990).

Finally, this resource text advocates strongly that educational research is not an end in itself. The intent underlying this summary of research results is that it should lead to some kind of positive action that improves mathematics education in Washington State. Suydam and Weaver (1975) issue the present challenge to both mathematics teachers and administrators: "You decide to change, or not to change; you will accept something, you will reject something…. **Do** something as a result of research: incorporate the conclusions of research [as tempered by unique attributes of your own situation and circumstances] into your daily teaching" (p. 6). Mathematics teachers and administrators who accept this challenge are taking big steps toward shifting from the "Yesterday Mind" to the "Tomorrow Mind."

Appendix A

FURTHER EXPLORATIONS

Bridging Educational Research and Teaching Mathematics:

English, L. and Halford, G. *Mathematics Education: Models and Processes*, Mahwah (NJ): LEA, 1995.

Fennema, E., Carpenter, T. and Lamon, S. (eds.) *Integrating Research on Teaching and Learning Mathematics*. Albany (NY): SUNY Press, 1991.

Grouws, D. (ed.) *Handbook of Research on Mathematics Teaching and Learning*. New York: MacMillan, 1992.

Jensen, R. (ed.) *Research Ideas for the Classroom: Early Childhood Mathematics*. Reston(VA): NCTM, 1993.

Leinhardt, G., Putnam, R. and Hattrup, R. *Analysis of Arithmetic for Mathematics Teaching*. Hillsdale (NJ): LEA, 1992.

Owens, D. (ed.) *Research Ideas for the Classroom: Middle Grades Mathematics*. Reston (VA): NCTM, 1993.

Wilson, P. (ed.) *Research Ideas for the Classroom: High School Mathematics*. Reston (VA): NCTM, 1993.

Alternative Assessments:

Gifford, B. and O'Connor, M. (eds.) *Changing Assessments: Alternative Views of Aptitude, Achievement and Instruction*. Norwell (MA): Kluwer, 1992.

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Information on National and International Assessments
    http://NCES.ed.gov

National Council of Teachers of Mathematics
    http://www.nctm.org

Office of Superintendent of Public Instruction (OSPI) Home Page
    http://www.k12.wa.us/
    NOTE: Contains links to many educational sites.

OSPI: Assessment Office
    http://assessment.ospi.wednet.edu

Washington State Mathematics Council
    http://wsmc.net

Eisenhower National Clearinghouse for Mathematics and Science
    http://www.enc.org

Ask ERIC: Educational Resources Information Center
    http://ericir.syr.edu

Science and Mathematics Consortium for Northwest Schools
    http://www.col-ed.org/smcnws/

Northwest Regional Educational Laboratory (NWREL)
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Appendix B

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