Listening to and Learning from Student Thinking

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Eliciting, responding to, and advancing students' mathematical thinking all lie at the heart of great teaching (NCTM 2014). We describe a formative assessment approach that teachers can use to learn more about their students' mathematical thinking and inform their instructional decisions. This assessment approach draws on a widely known set of frameworks for children's thinking, Cognitively Guided Instruction (CGI) (Carpenter et al. 2014). It enables teachers to learn from students by giving them time to voice their understandings and confusions. By listening carefully, teachers convey to students that their experience matters.

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2016 Focus Issue:
REVEALING THE
FACETS
OF ASSESSMENT

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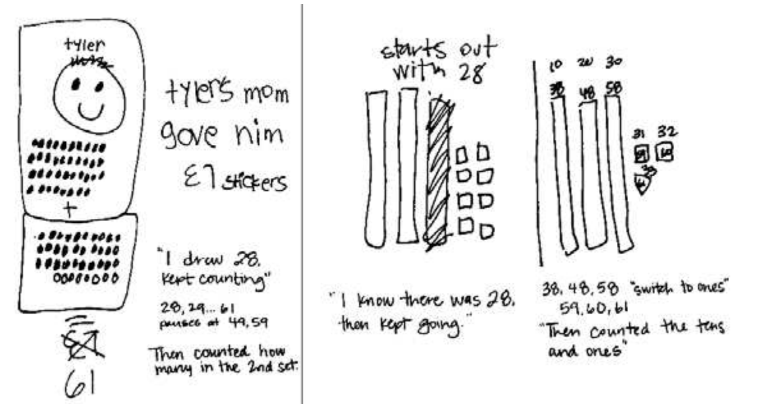
Learn how teachers can use assessment data coupled with observing students' mathematical reasoning to inform instructional decisions.

FIGURE 1

Yahye's work on a join-change-unknown problem and Brown's notes about his thinking show Yahye's improved understanding from the beginning to the end of the school year.



Gathering data to inform instruction

Data-driven decision making has become a major pillar of policies aimed at school improvement and consequential in efforts to achieve equitable outcomes for students. Teachers and school leaders are asked to use data to inform teaching practices and their plans for intervention, remediation, and acceleration. *Principles to Actions: Ensuring Mathematical Success for All* (NCTM 2014) points to the importance of using assessments to monitor students' progress and make instructional decisions to facilitate learning. Assessment results, however, may not convey to teachers how students think about problems, what they grapple with, and what approaches they take. And assessment results can leave teachers wondering about the nature of students' insights as well as their errors. For example, students may arrive at the wrong answer despite sound reasoning. We wanted a way to gain insight into student thinking that was manageable as well as communicated our deep respect for students' ideas and could help teachers make well-informed decisions about instruction. To this end, the authors, a team of teacher educators, worked closely with teachers at a local school to develop an assessment approach that could be implemented school-wide and involved face-to-face conversations with students. The assessment approach we developed emerged from a school-university partnership.

We begin by describing one teacher's experience with posing problems, watching her students work on them, and then collaborating with her school mathematics coach to make sense of her observations. (The vignette and data described in this article are based on our interactions with teachers and assessment data from local schools.) Ms. Brown, a second-grade teacher, gives her class four problems, including a missing-addend problem (called *join-change-unknown* in CGI frameworks) and circulates among her students to confer with them. She stops at Yahye's desk as he solves the following problem:

> Tyler had 28 stickers. His mom gave him some more stickers for his birthday. Now Tyler has 61 stickers. How many stickers did his mom give him?

Brown notices that Yahye has incorrectly answered, "Thirty-seven" (see **fig. 1**). She asks him to explain what he did and listens carefully to see if he understood the question, had a strategy to solve the problem, and made a computational error or lost track somewhere. She makes notes in the margins of his paper.

"First I drew twenty-eight stickers for Tyler. Then I needed some more, so I kept counting," Yahye explains.

Brown asks Yahye to show her how he counted. He touches his pencil to each circle as he answers, "Twenty-nine, thirty, thirty-one…"

Brown notices that Yahye hesitates momentarily at the end of some decades—from 49 to 50 and from 59 to 60—as he thinks about the name for the next number. She knows that English is Yahye's third language and that the pauses reflect his progress in learning the decade names. The brief hesitations, however, cause him to lose the rhythm of touching one circle for each number, which accounts for his error.

By listening to Yahye explain his thinking, Brown learns that he is trying to model the situation in the problem by first drawing the amount Tyler had and then adding more, one at a time, until he reaches 61 total stickers—an appropriate strategy for early in second grade (CCSSM 2010). Brown also notices some emerging ideas about place value in the way Yahye organizes twenty-eight into two rows of ten and one row of eight. Although Yahye's answer is wrong, his explanation shows important mathematical insights as well as potential next steps for his learning. Brown is already thinking about how to help him use the way he organizes his rows to count by tens as well as by ones.

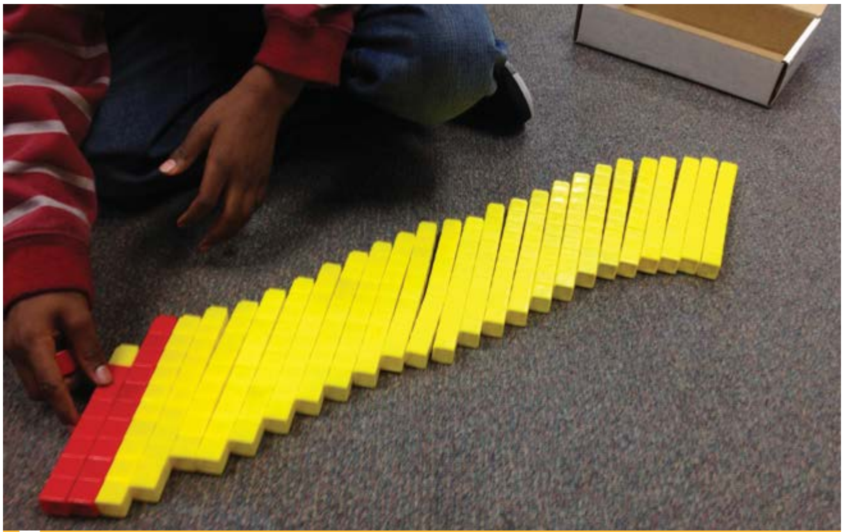
In designing our assessment approach, we sought to gain insight into students' ideas about important topics in elementary school

TABLE 1

This sample of story problems was used with elementary school students. Possible number size is in parentheses. Problem types, contexts, and number size can be adjusted as necessary.

Grade	Task	Skills we looked for
K–1	Counting and representing a set of objects; 30 cubes for kindergarten; 65 cubes for grade 1	• Counting sequence • Cardinality • One-to-one correspondence • Development of verbal, quantitative, symbolic correspondences for number
	Problem	**Strategies we looked for**
K–1	**Join-result-unknown** *Sumaya had 5 stickers. Her mom gave her 7 more stickers. How many stickers does she have in all?*	• Direct model • Count on • Derived facts • Recall
2–5	**Join-change-unknown** *Tyler had 28 stickers. His mom gave him some more stickers for his birthday. Now Tyler has ___ stickers. How many stickers did his mom give him?* (Early grade 2: 61) (Late grade 2–5: 111)	• Direct model by ones or by tens • Count on by ones or by tens • Invented algorithms • Standard algorithm
K–5	**Multiplication** *Zoey had ___ packs of pencils. There are ___ pencils in each pack. How many pencils does she have?* (K–grade 1: 3, 5) (Grades 2–3: 5, 12) (Grades 4–5: 8, 12)	• Direct model by ones or by tens • Skip count • Invented algorithms • Standard algorithm • Recall
2–5	**Division** *Ahmed has ___ pieces of candy. He wants to put 10 candies in each box. How many boxes will he need? How many candies will be left?* (Grades 2–3: 146) (Grades 4–5: 246)	• Direct model by ones or by tens • Skip count • Invented algorithms • Standard algorithm • Direct place-value understanding
1–5	**Fair-sharing** *2 children share 5 cookies evenly* (K–grade 1) *6 children share 10 cookies evenly* (Grades 2–5)	• Partitioning strategies • Representation • Language • Notation
1–5	**Relational thinking and understanding of equality** *9 + 3 = ___ + 5* (Grades 1–5) *45 – 26 + 25 = ___* (Grades 2–5)	• Understanding the equal sign to mean "the same as" • Ability to evaluate the whole number sentence

mathematics. CGI (Carpenter et al. 2014) informs the problem types and number sizes we use (see **table 1**). The items are not meant to provide a comprehensive assessment of student learning on all topics and domains or to assess particular grade-level standards. Instead, they provide an entry point into students' thinking about some key ideas about number and operations. Teachers select numbers strategically, those that allow them to see how students cross decades or centuries or use ideas about place value. For example, posing a division context with ten as the divisor helps teachers see how students work with the idea of ten as a unit. Division is not formally included in the second-grade Common Core State Standards (CCSSM 2010), but this item allows teachers to use a problem context to see how students reason about how many tens are in a given number (content standard 2.NBT.A.1). Including a range of problem types also enables teachers to see whether students are making sense of problem



ELHAM KAZEMI

The teacher has tools readily available for students to use as needed to solve problems in ways that make sense to them.

situations or if they are using “tricks” to solve problems, such as using key words to choose an operation. With kindergarten and first-grade students, teachers typically pose a smaller set of problems but also watch students count and record a collection of items (see **table 1**). The teachers we work with gather these data across grades at least twice a year.

How data are gathered is important

To create an environment in which students feel comfortable voicing their understandings and confusions, teachers make every attempt to communicate that they are interested in their students’ ideas and encourage students to solve the problems in a way that makes sense to them. Teachers make tools available—like hundred charts, Unifix® cubes, and base-ten blocks—for students to use as needed. As students work, teachers read the story aloud when appropriate, ask students to explain their thinking, and record students’ verbal explanations on the paper right alongside their work. Documenting their students’ thinking helps teachers remember how students solved the problem and simultaneously signals to students that their ideas are taken seriously and that the teacher is listening carefully. If students hesitate to get started, or they express confusion, the following questions help teachers better understand students’ uncertainty:

- “What’s confusing about this problem?”
- “Can you tell me what this story is about?”
- “Do you know what ____ means?”

To collect data, teachers work with support staff and specialists to confer with students individually or in small groups.

The CGI frameworks (Carpenter et al. 2014) provide a common language for describing the strategies that students use for each problem and for understanding how those strategies progress. For the join-change-unknown, multiplication, and division problems, strategies that students develop follow a similar trajectory. Students begin with modeling strategies, representing and counting all quantities. With experience, they learn efficient counting strategies then finally use invented strategies, flexibly applying their knowledge of place value and properties of operation. (See **fig. 2** for an illustration of this trajectory with the join-change-unknown problem; to read more about students’ strategies for solving the fair-sharing item, see Lewis et al. 2015.) When students use a standard algorithm to solve a problem, teachers prompt them to use an additional strategy to better understand the depth of their repertoire of strategies. By watching students solve these problems, teachers can see (1) whether students are making sense of problems, (2) the types of strategies they use to solve the problem, and (3) whether they are accurately using the strategy.

Analyzing student work

With a snapshot of each student’s thinking, Brown can analyze her students’ mathematical understanding. Let’s look at four additional student work samples from Brown’s class for the missing addend problem (see **fig. 2**).

Examining student work across a class

Jazmin uses a direct-modeling strategy (see **fig. 2a**), as Yahye did. She draws the initial 28 stickers and then more stickers, one by one, until she gets to 111. Counting and drawing individual squares is quite laborious. Brown notices that in the process of drawing each square, Jazmin draws 112 instead of 111. When Jazmin recounts the number of stickers she added, she counts 84 because of that extra sticker.

Camron’s counting strategy is more efficient because he starts at 28 and counts on more stickers by adding groups of 10 until he reaches 108. He then counts on by ones to reach 111. To get his answer, he goes back to count the stickers he added on (see **fig. 2b**).

Brown notices that Samira’s invented strategy makes use of what she knows about number composition and the counting sequence to add incrementally in groups of 30 and 10 to get from 28 to 108 (see **fig. 2c**). Some students use invalid strategies that do not fit the situation. One example is Lyndon's strategy of simply adding the two numbers together: 28 + 111 (see **fig. 2d**).

After school, Brown works with her school mathematics coach to compile the accuracy and strategy data for her class (see **table 2**). They notice that the accuracy rates are quite low across all the problems. However, some students used strategies that should have led to an accurate answer. For example, Brown notes, "Eighteen students used a valid strategy for the missing addend problem, but only ten answered the problem accurately. I wonder what's going on when they get it wrong."

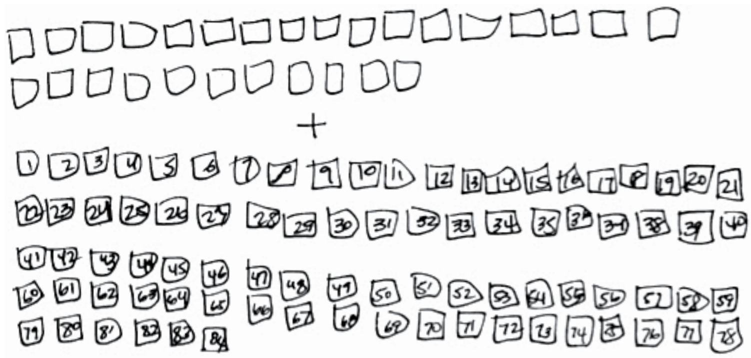
Together, Brown and the coach continue investigating where students made errors and why. They also observe that students did not use invented strategies to solve the multiplication or division problems. This discovery prompts them to make plans to help students decompose numbers more flexibly and work toward invented algorithms that make use of our base-ten structure and the meaning of the operations. Finally, they notice that only five students correctly interpreted the equal sign as "the same as." As students develop numerical methods to record their strategies, the teachers want students to have an appropriate understanding of the equal sign. The coach remarks,

FIGURE 2

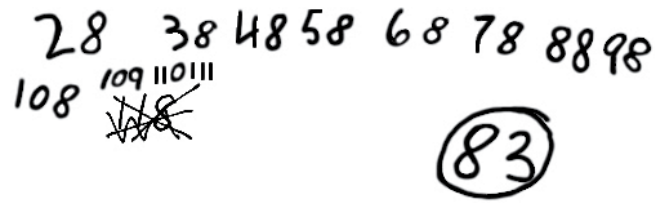
The student work samples below are for a join-change-unknown problem:

Tyler had 28 stickers. His mom gave him some more stickers for his birthday. Now Tyler has 111 stickers. How many stickers did his mom give him?

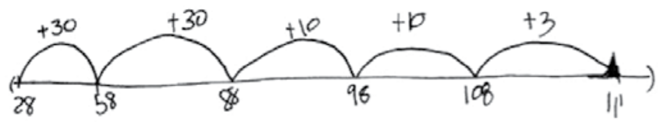
(a) Jazmin: Direct modeling by ones



(b) Camron: Counting on by tens



(c) Samira: Invented algorithm: Incrementing



(d) Lyndon: Invalid strategy

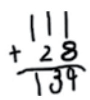


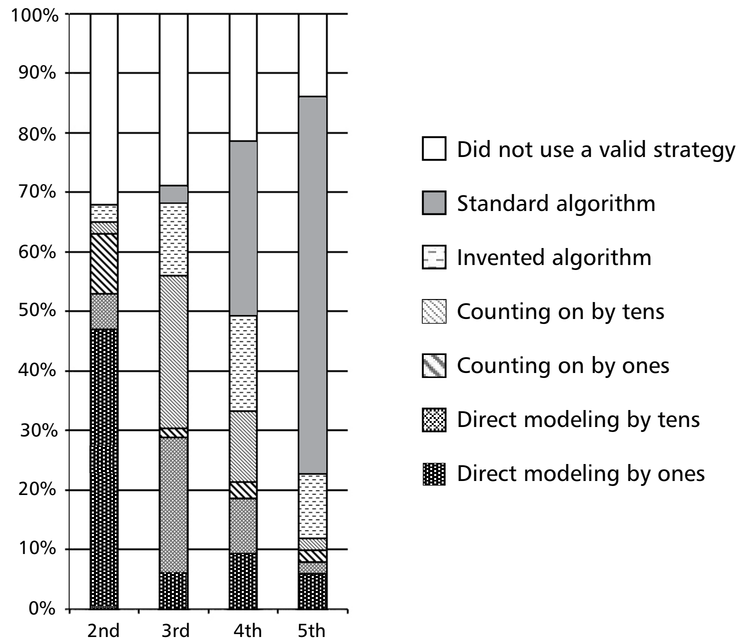
TABLE 2

This table show strategies that students used in Brown's class.

Problem	Number correct (of 26)	None	Type of strategy: Direct modeling	Type of strategy: Counting	Type of strategy: Invented algorithm
Join-change-unknown (28 + ___ = 61)	10	8	13	4	1
Multiplication 5 × 12	16	10	12	4	0
Division 146 ÷ 10	12	9	6	11	0
		Answered 12 or 17		Answered 7	
Meaning of equality 9 + 3 = ___ + 5	5	21		5	

FIGURE 3

On the basis of the percentage of students using each strategy for a join-change-unknown problem across grades 2–5 in a school, all the teachers decided to help their students develop invented strategies (See **table 1** for problems and quantities given to students.)



KENDRA LOMAX

Investigating *where* students made errors and *why* can inform instructional decisions to resolve misconteptions.

"It seems like many students are reading the equal sign to mean 'put your answer here.' Let's try varying the format of equations as we record students' strategies so they don't always look like $a + b = c$ and see if that pushes them to think about equality a little more broadly."

By examining student work together, Brown and the coach are coming to better understand each student as a mathematician. These conversations also enable the coach to support Brown and other teachers in the school as they refine their own knowledge of students' learning trajectories and how best to respond pedagogically.

Examining data across classes and grade levels

Collectively examining student work can support educators' efforts across a school as well as fostering a shared sense of responsibility for improvement (NCTM 2014). For example, **figure 3** shows students' strategies on a join-change-unknown problem across grades 2–5 in Brown's school. In an after-school meeting, the coach shares data gathered from each grade. Teachers note that 32 percent of second-grade students and 28 percent of third-grade students did not use a valid strategy to solve the problem. They notice that more students in fourth and fifth grade solved the problem using a valid strategy, but more than 60 percent of fifth-grade students used only a standard algorithm and did not show flexibility in using other efficient strategies. On the basis of these insights, teachers across all grades decide to focus on helping students develop a broader range of invented strategies that show knowledge of place value and the meaning of the operations.

Together, teachers and school leaders at Brown's school consider which types of instructional activities to use across grade levels to develop students' number sense and computational fluency, tailoring these activities to the particular learning needs of individual students and classes. Brown knows that current testing practices often encourage schools to sort and label students by ability and can therefore constrain students' learning (DiME 2007). She is careful to discourage teachers from deciding that they must put students into like-strategy groups. Coaches and principals can help teachers interpret and respond to data by using tasks and instructional practices that allow for multiple entry points and improve the quality of students' learning. After looking across their data, primary grade teachers decide to implement such classroom activities as Counting Collections (Schwerdtfeger and Chan 2007) or Choral Counting (see TEDD.org) to improve students' counting strategies and place-value ideas. In the intermediate grades, teachers decide to implement activities such as Number Strings (Parrish 2010; Humphreys and

Parker 2015) or True/False number sentences (Carpenter, Franke, and Levi 2003) to help students develop more efficient strategies and understanding of equality.

Examining student understanding over time

By paying attention to students' strategies and using learning trajectories such as those described in the CGI frameworks, teachers can track how students develop more sophisticated understandings (NCTM 2014). For example, see Yahye's work on the same join-change-unknown problem given to him again later in the school year (see **fig. 1**). In the spring, he draws twenty-eight using long rectangles to indicate groups of ten and smaller squares to show ones. This time Yahye counts on by tens as he draws more rectangles, switching to counting by ones when he reaches fifty-eight. Yahye's use of tens to organize stickers, rather than counting individual ones as he did in the fall, indicates the growth in his understanding of tens and ones. Information like this, across multiple problem types and across time, can provide a detailed picture of what students know about number and operations and can suggest next steps for learning. In this case, Brown continues to encourage Yahye to move toward counting on by tens, by exploring whether he is ready to solve the problem without making the initial set of twenty-eight.

Conclusion

The primary purpose of assessment is to inform and improve the teaching and learning of mathematics (NCTM 2014). The assessment approach we describe in this article gives teachers opportunities to talk with students about their mathematical thinking and gain a better understanding of their students as mathematicians. The information gathered provides rich material for teachers to consider what to do next to support student learning. Importantly, this approach also allows teachers to convey to students that their mathematical ideas matter. How we gather data about and with students is important for equity. In a time when high stakes tests determine who is making it in our current systems, we need to make stronger efforts to hear our students and to give them voice.

MATT HAGEN

After analyzing students' strategies across grade levels, teachers developed a plan to guide all students toward a broader range of invented strategies.

Common Core Connections

1.NBT.1
2.OA
4.NBT.1

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